

# The Lemma on $b$ -functions in Positive Characteristic

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## Abstract

Let  $X$  be an  $F$ -finite smooth scheme of essentially finite type over a perfect field. This article proves the existence of  $b$ -functions for locally finitely generated unit  $F$ -modules when equipped with their induced  $\mathbb{D}_X$ -module structure. It is shown that the  $b$ -function can be chosen to have rational roots and is determined locally in the étale topology.

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## 1 Introduction

The purpose of this article is to define the notion of a  $b$ -function in positive characteristic, prove one exists locally, and prove that it can be chosen to have rational roots. More pre-

cisely, we define the notion of a  $b$ -function for generators of unit  $F^n$ -modules. We show when  $X$  is a smooth variety over a perfect field and the generator is coherent, then the  $b$ -function exists locally and has rational roots. The definition of the  $b$ -function proposed in this article was inspired by M. Mustață's analysis [Mus09] of the relationship between  $F$ -jumping exponents for the test ideals  $\tau(f^\lambda)$  and the actions of the higher Euler operators on the first local cohomology module along the graph of  $f$ . The definition presented in this paper depends on (locally) choosing a generating morphism in the sense of G. Lyubeznik [Lyu97].

**Definition** Let  $X = \text{Spec}(R[t])$  and  $A : M \rightarrow F_X^{n*}M$  be a morphism generating the unit  $F^n$ -module  $(\mathcal{M}, F^n)$ . A  $b$ -function for  $\mathcal{M}$  is a polynomial  $b(s) \in \mathbb{R}[s]$  and an integer  $N$  with the following property for every integer  $e > 0$ : If  $\lambda$  is a root of  $b(s - 1)$  then  $\lambda \in (0, 1]$  and for every  $0 \leq a < q^N$  if  $\lceil \lambda q^{e+1} \rceil - a = j_0(\lambda, a) + j_1(\lambda, a)p + \dots + j_{ne}(\lambda, a)p^{ne}$  is the base  $p$  expansion, then

$$\prod_{0 \leq a < q^N} \prod_{\substack{\lambda \\ b(\lambda - 1) = 0}} -\Theta_i + 1 + j_i(\lambda, a) \text{ annihilates } \mathbb{D}_R^e[t, \theta_1, \dots, \theta_e]M / \mathbb{D}_R^e[t, \theta_1, \dots, \theta_e]tM$$

where  $\Theta_i = \partial_t^{[p^{i-1}]} t^{p^{i-1}}$ .

The higher Euler operators satisfy the simple equation  $\Theta_i^p - \Theta_i = 0$  but it is not obvious that the  $b$ -function exists in any non-trivial cases. A priori, for each  $e$  the definition could require the use of  $p^{ne-N+1}$  distinct  $\lambda$ . The first non-trivial case was discovered by M. Mustață in [Mus09] where it was proven that a  $b$ -function for the first local cohomology module along the graph of  $f$  is the polynomial  $b(s) = \prod_i (s + \lambda_i)$  where  $\{\lambda_i\}$  are the  $F$ -jumping exponents of  $f$  in  $(0, 1]$ . This is a very close analogue to the analytic setting where the roots of the  $b$ -function are related to the filtration by multiplier ideals. The work in [Mus09] also implies, since  $F$ -jumping exponents are rational numbers, that  $b(s)$  has rational roots. The main result of [Mus09] is generalized as the main theorem of this paper.

**Main Theorem (Ring-theoretic version).** If  $R$  is a commutative  $F$ -finite ring which is smooth and of essentially finite type over a perfect field  $\mathbb{k}$ ,  $A : M \rightarrow F_X^{*n}M$  a root morphism generating a unit  $F^n$ -module  $(\mathcal{M}, F^n)$  on  $R[t]$  with  $M$   $R[t]$ -finite, then it admits a  $b$ -function with rational roots.

There is also the algebro-geometric analogue.

**Main Theorem (Algebro-geometric version).** If  $X$  is a smooth  $F$ -finite scheme of essentially finite type over a perfect field  $\mathbb{k}$ ,  $(\mathcal{M}, F^n)$  a unit  $F^n$ -module on  $X \times \mathbb{A}^1$ , then for every affine open set  $U \subset X$ , every  $\mathcal{O}_{U \times \mathbb{A}^1}$ -coherent generator  $(M, A)$  of  $(\mathcal{M}|_{U \times \mathbb{A}^1}, F^n|_{U \times \mathbb{A}^1})$  admits a  $b$ -function with rational roots. The  $b$ -function is stable under restriction to étale neighborhoods.

Before introducing some background, let us first mention some important and motivating examples.

**Example.** If  $R$  is a commutative regular  $F$ -finite ring of essentially finite type over a perfect field  $\mathbb{k}$  and

$$\mathcal{M} = R[t]_{(f-t)}/R[t] \cong H_{\Gamma_f}^1(R)$$

with standard  $F$ -structure given by  $\bar{h} \mapsto \bar{h}^p$ , then a  $b$ -function for the generating morphism

$$R[t]/(t) \rightarrow R[t]/(t) \quad \bar{h} \mapsto \overline{h(f-t)^{p-1}}$$

is given by the polynomial whose roots are the negatives of the  $F$ -jumping exponents for the generalized test ideals  $\tau(f^\lambda)$ . This statement will be proven in this paper but is originally contained in [Mus09] in a stronger form.

**Example.** If  $R$  is a commutative regular  $F$ -finite ring of essentially finite type over a perfect field  $\mathbb{k}$ ,  $p^n = 1 \pmod{m}$ , and

$$\mathcal{M} = \mathbb{D}_{R[t]} \sqrt[m]{t}$$

with standard  $F^n$ -structure given by  $h \mapsto h^{p^n}$ , then a  $b$ -function for the generator  $\sqrt[m]{t}$  is given by  $(s + \frac{1}{m})$ . This is the same as the  $b$ -function (using operator  $-\partial_t$ ) in characteristic zero.

**Example.** If  $R = \mathbb{k}[t]$  with  $\mathbb{k}$  a perfect field and

$$\mathcal{M} = \mathbb{k}[t, t^{-1}, u]/(u^{p^n} + tu^{p^n-1} - t)$$

with standard  $F^n$ -structure arising from being the push forward of a local system on  $\mathbb{k}[t, t^{-1}]$ , then a  $b$ -function for the canonical generator is given by

$$\prod_{0 \leq m < p^n} (s + \frac{m}{p^n}).$$

Let us now briefly summarize the theory of  $b$ -functions in characteristic zero so that it may be related to the definition given in positive characteristic. Recall that when  $X$  is a smooth complex algebraic variety with  $Z \subset X$  a smooth hypersurface defined globally by the sheaf of ideals  $\mathcal{I}_Z$ , the ring of differential operators  $\mathbb{D}_X$  can be naturally filtered by setting

$$V^i \mathbb{D}_X = \{P \in \mathbb{D}_X \mid P \mathcal{I}_Z^j \subset \mathcal{I}_Z^{j+i} \forall j\}.$$

When  $X = \text{Spec}(S)$  with  $S = \mathbb{C}[x_1, \dots, x_n, t]$  and  $Z = \{t = 0\}$ , then  $\mathbb{D}_X$  is the Weyl algebra on  $n+1$  symbols,  $\mathbb{C}\langle x_1, \dots, x_n, t, \partial_1, \dots, \partial_n, \partial_t \rangle$ . The  $V$ -filtration is given by placing  $x_1, \dots, x_n, \partial_1, \dots, \partial_n$  in degree 0,  $t$  in degree 1 and  $\partial_t$  in degree  $-1$ . The 0-th component of the associated graded to this filtration is  $\mathbb{D}_Z[t\partial_t]$ . For the benefit of the reader, the remainder of this introduction will be restricted to this case.

Given a finitely generated  $\mathbb{D}_X$ -module,  $\mathcal{M}$ ,  $j \in \mathbb{Z}$ , and an element  $M \subset \mathcal{M}$ , we can consider the  $\mathbb{D}_Z[-\partial_t t]$ -module  $N_{M,j} = V^j \mathbb{D}_X M / V^{j+1} \mathbb{D}_X M$ . A  $b$ -function for  $\mathcal{M}$  is said to exist if for all  $\mathcal{O}_X$ -coherent sets  $M \subset \mathcal{M}$  generating  $\mathcal{M}$  as a  $\mathbb{D}_X$ -module, there is a non-zero function  $a_M(s) \in \mathbb{C}[s]$  such that  $a_M(-\partial_t t - j)$  annihilates  $N_{M,j}$  for all  $j$ . The minimal degree monic polynomial with this property is denoted by  $b_M(s)$ . As the filtration  $V_Z^i \mathbb{D}_X$  is well-behaved in characteristic 0, it turns out that a  $b$ -function exists for  $\mathcal{M}$  if and only if for every  $M$  there is a polynomial  $c_M(s)$  such that  $c_M(-\partial_t t)$  annihilates  $N_{M,0}$ . For instance, if for every  $M$   $N_{M,0}$  is holonomic as a  $\mathbb{D}_Z$ -module, then  $s \mapsto -\partial_t t$  determines a map  $\mathbb{C}[s] \rightarrow \text{End}_{\mathbb{D}_Z}(N_{M,0})$ . The latter ring is a finite dimensional vector space, thus  $a_M(s)$  exists for all  $M$  and  $\mathcal{M}$  admits a  $b$ -function. More generally, it is known that if  $\mathcal{M}$  is regular and holonomic, then a  $b$ -function for  $\mathcal{M}$  exists.

Given a regular function  $f : \mathbb{A}_{\mathbb{C}}^n \rightarrow \mathbb{C}$  with an isolated singularity at the origin, one can consider the  $\mathbb{D}_{\mathbb{A}_{\mathbb{C}}^{n+1}}$ -module  $i_f^* \mathcal{O}_{\mathbb{A}_{\mathbb{C}}^n}$  where  $i_f$  is the inclusion of the graph. Inside of this module there is a canonical generating element  $\delta$  and it turns out that very interesting things can be said about  $b_\delta(s)$ . It is known that  $b_\delta(s)$  exists, has rational roots, and encodes information about the singularities of  $f$ .

In the positive characteristic setting, many of the same questions may be posed. There are two major difficulties. The first is that the  $V$ -filtration as defined above no longer behaves very well. The second is that the Euler operator  $-\partial_t t$  satisfies the simple equation  $x^p - x = 0$ , making all statements trivial. An equivalent condition to the action  $-\partial_t t$  satisfying a polynomial equation on  $V^0 \mathbb{D}_X M / V^1 \mathbb{D}_X M$  is the requirement that  $V^0 \mathbb{D}_X M / V^1 \mathbb{D}_X M$  is a finite direct sum of generalized eigenspaces for the action of  $-\partial_t t$ . Motivated by this and using the higher Euler operators  $\partial_t^{[n]} t^n$ , in positive characteristic we could simply ask for any  $\mathcal{O}_X$ -coherent generating set  $M$  that  $V^0 \mathbb{D}_X M / V^1 \mathbb{D}_X M$  breaks up as a finite direct sum of multi-eigenspaces for the entire collection  $\{-\partial_t^{[n]} t^n\}$ . It was observed in [Mus09] that this condition is too weak to capture all of the interesting behavior of the action of the higher Euler operators on the first local cohomology module along the graph of a function. The more interesting object of study in [Mus09] was the quotient  $V^0 \mathbb{D}_X^e M / V^1 \mathbb{D}_X^e M$  where  $\mathbb{D}_X^e$  is the  $p$ -filtration. Thus we set out the following criterion for the existence of a  $b$ -function.

### Criterion for a $b$ -function

Fix  $M \subset \mathcal{M}$  where  $\mathcal{M}$  is a  $\mathbb{D}_X$ -module generated by  $M$  with  $M$   $\mathcal{O}_X$ -coherent. It is trivial to observe that  $V^0 \mathbb{D}_X^e M / V^1 \mathbb{D}_X^e M$  is a direct sum of multi-eigenspaces for the collection  $\{-\partial_t^{[p^{i-1}]} t^{p^{i-1}}\}_{i=1}^e$ . The main conditions we would like to require are:

1. (Finiteness) The number of (non-zero) multi-eigenspaces of  $V^0 \mathbb{D}_X^e M / V^1 \mathbb{D}_X^e M$  is uniformly bounded (independent of  $e$ ).
2. (Relationship) For any  $e$  and  $k$ , the multi-eigenspaces of  $V^0 \mathbb{D}_X^{e+k} M / V^1 \mathbb{D}_X^{e+k} M$  are related to the multi-eigenspaces of  $V^0 \mathbb{D}_X^e M / V^1 \mathbb{D}_X^e M$ .

This criterion helped to guide the definition of the  $b$ -function given in the first paragraph. However, we also hope to prove its existence. In the complex setting, it was shown to exist as long as the module was regular and holonomic. Therefore, in order to construct an existence theorem in positive characteristic it is necessary to restrict our category to one smaller than the category of all  $\mathbb{D}_X$ -modules. The work of M. Emerton and M. Kisin [EK04] suggests a setting in which the  $b$ -function may exist. In [EK04], it was shown that the category of locally finitely generated unit  $F_X^n$ -modules is equivalent to the category of perverse  $\mathbb{F}_{p^n}$ -sheaves on the étale site of  $X$ . This is an analogue of the complex case where the category of regular holonomic  $\mathbb{D}_X$ -modules is equivalent to the category of perverse  $\mathbb{C}$ -sheaves on the analytic site of  $X$ . Thus, a natural setting in which to search for  $b$ -functions is within the context of locally finitely generated unit  $F^n$ -modules. In terms of this setting, we should weaken the investigation from generators of  $\mathcal{M}$  as a  $\mathbb{D}_X$ -module to coherent generators of  $\mathcal{M}$  as unit a  $F^n$ -module.

The approach to proving the main theorem of this paper is to construct a generalization of generalized test ideals which are proposed to be called “list test ideals” and “list test modules”. It is shown in 3.4 that one can use this construction to recover test ideals. With these “list test modules”, there is a way to define an analogue of  $F$ -jumping exponents, which are simply called jumping numbers. When generalized test ideals are constructed as list test ideals, the jumping numbers of the latter will coincide with the  $F$ -jumping exponents of the former. These jumping numbers share some of the same properties of  $F$ -jumping exponents: If  $\lambda$  is a jumping number then so are  $p\lambda$  and  $\lambda - 1$ , the set of jumping numbers is discrete, and every jumping number is rational. This paper also develops some further theory, namely exhibiting that the jumping numbers are étale local in nature. The purpose of the appearance of the jumping number associated to “list test modules” stems from a certain reduction of cases for proving the  $b$ -function exists. It is shown that one can reduce to the case where the coherent generator is free of finite rank and then a  $b$ -function is determined by a polynomial whose roots are the jumping numbers of some list test module. Hence, the  $b$ -function exists, has rational roots, and étale local in nature.

The layout of the paper is as follows. In the second section, some overview and unifying of standard notation is developed for use later in the paper. The third section is dedicated to the development of the theory of “list test modules”, the development of the theory of jumping numbers, and the proof that they are discrete and rational. The fourth section analyzes the action of the Euler operators, defines the existence of a  $b$ -function, reduces the case to the one mentioned in the previous paragraph. This section then relates the existence of the  $b$ -function in that case to the jumping numbers associated to certain “list test modules”.

## 2 Preliminary Information

This section will discuss notation and provide background on the structural properties of certain  $\mathbb{D}$ -modules, called unit  $F$ -modules. The results in this section are restatements, obvious generalizations, combinations, or immediate corollaries of results contained in [BMS09], [EK04], [Haa87], and [Lyu97].

### 2.1 Notation

**Notation 2.1.** Throughout this article the following conventions will be used.

- $\mathbb{k}$  is a (perfect) field of characteristic  $p$ .
- $q$  is a power of  $p$
- For a scheme  $X$ ,  $F_X$  is the (absolute)  $q^{th}$ -power Frobenius map on  $X$  given by  $f \mapsto f^q$ .
- $R$  is a regular Noetherian commutative  $\mathbb{k}$ -algebra and the map  $F_R$  is finite.
- $X$  is a regular scheme over  $\mathbb{k}$ , the map  $F_X$  is finite.
- $\mathbb{D}_X \subset \mathcal{E}nd_{\mathbb{k}}(\mathcal{O}_X)$  is Grothendieck's sheaf of differential operators.
- A map of  $R$ -modules,  $T : M \rightarrow N$ , is called  $q$ -linear if  $T(rm) = r^q T(m)$ .

**Remark 2.2.** The use of  $F_X$  to denote the  $q^{th}$ -power Frobenius is slightly non-standard. However, in this article it will be convenient to use a notation that allows us to increase the power of  $p$  without complicating the notation.

### 2.2 Unit $F_X$ -modules

Unit  $F$ -modules were investigated by G. Lyubeznik in [Lyu97] where many of their fundamental properties were proven. Most notably, he defined the notion of a generating morphism and an  $F$ -finite unit  $F$ -module. He also studied how  $F$ -finite unit  $F$ -module, which carry a natural structure of  $\mathbb{D}_X$ -module, had very special properties as  $\mathbb{D}_X$ -modules. We begin by stating a slight variation on the definition of unit  $F$ -modules given in [Lyu97].

**Definition 2.3.** A unit  $F$ -module is a pair  $(\mathcal{M}, F)$  of a quasi-coherent  $\mathcal{O}_X$ -module,  $\mathcal{M}$ , with a  $q$ -linear endomorphism  $F : \mathcal{M} \rightarrow \mathcal{M}$  such that the induced map  $\theta^{-1} : F_X^* \mathcal{M} \rightarrow \mathcal{M}$  defined locally by  $\theta^{-1}(f \otimes m) = fF(m)$  is an isomorphism.

There is also a corresponding variation of the construction given for creating unit  $F$ -modules in [Lyu97, 1.9].

**Definition-Construction 2.4.** Let  $M$  be a quasi-coherent  $\mathcal{O}_X$ -module and  $A : M \rightarrow F_X^* M$  a map of  $\mathcal{O}_X$ -modules. Iterating pull-back by the Frobenius map yields the directed system

$$M \xrightarrow{A} F_X^* M \xrightarrow{F_X^*(A)} F_X^{2*} M \xrightarrow{F_X^{2*}(A)} \dots$$

Let  $\mathcal{M}$  denote the direct limit of this directed system and  $\mu_e : F_X^{e*} M \rightarrow \mathcal{M}$  the inclusion maps. By construction,  $\mu_{e+1} \circ F_X^{e*}(A) = \mu_e$ . Define  $F$  to be the  $q$ -linear endomorphism of  $\mathcal{M}$  given by  $F(\mu_e(g \otimes m)) = \mu_{e+1}(g^q \otimes m)$ .

The pair  $(\mathcal{M}, F)$  is **the unit  $F$ -module generated by  $(M, A)$** .

$(M, A)$  is **a generating morphism for  $(\mathcal{M}, F)$** .

$(\mathcal{M}, F)$  is **locally F-finite or locally finitely generated** if locally it admits a generating morphism  $(M, A)$  with  $M$   $\mathcal{O}_X$ -coherent.

*Proof.* It needs to be confirmed that  $\mathcal{M}$  induces a unit  $F$ -structure on  $\mathcal{M}$ .

As observed in [Lyu97], there is a natural isomorphism

$$\theta : \mathcal{M} = \varinjlim_e F_X^{(e+1)*} M \rightarrow F_X^* \varinjlim_e F_X^{e*} M = F_X^* \mathcal{M}$$

defined locally by

$$\mu_{e+1}(f \otimes m) \mapsto f \otimes \mu_e(1 \otimes m).$$

We need to show that  $\theta^{-1}$  is as prescribed by the two previous definitions. To see this, note that

$$\theta(\mu_{e+1}(f g^q \otimes m)) = f g^q \otimes \mu_e(1 \otimes m) = f \otimes \mu_e(g \otimes m),$$

therefore

$$\theta^{-1}(f \otimes \mu_e(g \otimes m)) = \mu_{e+1}(f g^q \otimes m) = f \mu_{e+1}(g^q \otimes m) = f F(\mu_e(g \otimes m)).$$

□

**Remark 2.5.** A unit  $F$ -module can be regarded as a left module for the sheaf of non-commutative algebras  $\mathcal{O}_X\{F\}$  which is defined to be the quotient of the free  $\mathcal{O}_X$ -algebra on the letter  $F$  by the two-sided ideal  $(f^q F - F f)$ . It was shown in [EK04] that a unit  $F$ -module  $(\mathcal{M}, F)$  is locally finitely generated over the ring  $\mathcal{O}_X\{F\}$  if and only if it is locally generated by  $(M, A)$  in the sense of 2.4 for appropriately chosen  $M$  and  $A$ .

It will be useful to have a systematic method for changing the unit  $F$ -structure and generating morphisms when  $q$  is replaced by  $q^\gamma$ .

**Lemma 2.6.** If  $q' = q^\gamma$  is a power of  $q$  and  $F'_X$  the  $q'^{th}$ -power Frobenius then any unit  $F$ -module  $(\mathcal{M}, F)$  naturally determines a unit  $F^\gamma = F'$ -module structure on  $\mathcal{M}$  given by  $(\mathcal{M}, F' = F^\gamma)$ . If  $(M, A)$  is a generating morphism for  $(\mathcal{M}, F)$  then  $(M, F_X^{(\gamma-1)*}(A) \dots A)$  is a generating morphism for  $(\mathcal{M}, F')$ .

*Proof.* For the first statement, it is clear that  $F'$  is  $q'$ -linear because  $F$  is  $q$ -linear. It is necessary to check that

$$F_X'^* \mathcal{M} \rightarrow \mathcal{M} \text{ by } f \otimes m \mapsto f F'(m)$$

is an isomorphism. The proof proceeds by induction on  $\gamma$ . The result is clear for  $\gamma = 1$  and assume it holds for  $\gamma - 1$ . The following diagram establishes the proof for  $\gamma$ .

$$\begin{array}{ccccccc} F_X'^* \mathcal{M} & \xleftarrow{\cong} & F_X^* (F_X^{(\gamma-1)*} \mathcal{M}) & \xrightarrow{\cong} & F_X^* \mathcal{M} & \xrightarrow{\cong} & \mathcal{M} \\ \Psi & & \Psi & & \Psi & & \Psi \\ f \otimes m & \longleftarrow & f \otimes (1 \otimes m) & \longmapsto & f \otimes F^{\gamma-1}(m) & \longmapsto & f F(F^{\gamma-1}(m)) = f F^\gamma(m) \end{array}$$

For the second statement, it is clear that as  $\mathcal{O}_X$ -modules  $\mathcal{M} \cong \varinjlim_e F_X'^e M$ . It is only necessary to check that the  $q'$ -linear map induced on  $\mathcal{M}$  by  $F_X^{(\gamma-1)*}(A) \dots A$  is  $F'$ . If  $\mu'_e$  denotes the inclusion maps of  $F_X'^e M$  into  $\mathcal{M}$ , then the induced  $q'$ -linear map is

$$\mu'_e(f \otimes m) \mapsto \mu'_{e+1}(f^{q'} \otimes m).$$

The  $\gamma^{th}$  composition of the  $q$ -linear map  $F$  induced by  $(M, A)$  on  $\mu_{\gamma e}$  is

$$\mu_{\gamma e}(f \otimes m) \mapsto \mu_{\gamma e + \gamma}(f^{q'} \otimes m).$$

By definition,  $\mu'_e = \mu_{\gamma e}$  and  $\mu'_{e+1} = \mu_{\gamma e + \gamma}$  so former induced map is  $F'$ . □

## 2.3 $\mathbb{D}_X$ -actions on unit $F$ -modules

We will now begin the investigation into the relationship between unit  $F$ -modules and  $\mathbb{D}_X$ . While this relationship is mentioned in [Lyu97], there is a more general and categorical approach presented in [Haa87]. The approach of the latter reference is used in this article to illustrate the category of unit  $F$ -modules as a full subcategory of “periodic”  $\mathbb{D}_X$ -modules.

**Remark 2.7.** It was shown in [Haa87, 1.2.5] that  $\mathbb{D}_X \cong \varinjlim_e \mathcal{E}nd_{q^e}(\mathcal{O}_X)$  where  $\mathbb{D}_X^e = \mathcal{E}nd_{q^e}(\mathcal{O}_X)$  denotes sheaf of abelian group endomorphisms,  $T$ , with  $T(f^{q^e} g) = f^{q^e} T(g)$  for all  $f \in \mathcal{O}_X$ . As  $X$  is smooth,  $\mathcal{E}nd_{q^e}(\mathcal{O}_X) \cong \mathcal{E}nd_{\mathcal{O}_X}(F_*^e \mathcal{O}_X)$  is a trivial Azumaya algebra and via the Morita equivalence, the category of left modules of this ring is equivalent to the the category of left  $\mathcal{O}_X$ -modules.



The previous remark implies that we may view a left  $\mathbb{D}_X$ -module,  $\mathcal{M}$ , as a quasi-coherent  $\mathcal{O}_X$ -module with compatible actions of the trivial Azumaya algebra  $\mathbb{D}_X^e$ . Thus, for each  $e$  we can apply the Morita functor to obtain a left  $\mathcal{O}_X$ -module  $\mathcal{M}_e$  with certain compatibility properties in terms of  $e$ . Therefore, a left  $\mathbb{D}_X$ -module may be viewed as a system of  $\mathcal{O}_X$ -modules with compatibility conditions. The following theorem from [Haa87] makes this process precise.

**Theorem 2.8.** The category of left  $\mathbb{D}_X$ -modules is equivalent to the category of diagrams.

$$\dots \rightarrow \mathcal{M}_{i+1} \xrightarrow{\phi_i} \mathcal{M}_i \rightarrow \dots \rightarrow \mathcal{M}_0$$

subject to the following conditions:

1.  $\mathcal{M}_{i+1}$  is a quasi-coherent sheaf on  $X$ .
2.  $\phi_i$  is  $q$ -linear.
3.  $\phi_i$  induces an isomorphism  $F_X^* \mathcal{M}_{i+1} \rightarrow \mathcal{M}_i$  defined locally by  $f \otimes m \mapsto f\phi_i(m)$ .

**Remark 2.9.** While a proof will not be presented, it will be useful to discuss how from such a diagram we may construct a  $\mathbb{D}_X$  action on  $\mathcal{M}_0$ . Given  $P \in \mathbb{D}_X^e$  and  $m \in \mathcal{M}_0$ , consider the given isomorphism  $\xi^{-1} : F_X^{e*} \mathcal{M}_e \rightarrow \mathcal{M}_0$ .  $F_X^{e*} \mathcal{M}_e$  carries a natural action of  $\mathbb{D}_X^e$  and we define  $Pm = \xi^{-1} P \xi(m)$ . The convention in the labeling of  $\xi^{-1}$  is used to be compatible with the notation in [Lyu97].

**Corollary 2.10.** A unit  $F$ -module,  $(\mathcal{M}, F)$ , on  $X$  has the natural structure of a  $\mathbb{D}_X$ -module.

*Proof.* Consider the diagram,

$$\dots \xrightarrow{F} \mathcal{M} \xrightarrow{F} \mathcal{M} \xrightarrow{F} \dots \xrightarrow{F} \mathcal{M}$$

and use the previous theorem. □

**Remark 2.11.** If  $q'$  is a power of  $q$  then changing the  $q$ -structure on a unit  $(\mathcal{M}, F)$ -module to a  $q'$ -structure as in 2.6 results in the same  $\mathbb{D}_X$ -structures on the module  $\mathcal{M}$ .

**Question:** If  $(\mathcal{M}, F)$  is generated by  $(M, A)$  then  $\mathcal{M}$  is a directed limit in terms of  $M$ . Can the action of  $\mathbb{D}_X$  on  $\mathcal{M}$  be understood in terms of  $(M, A)$ ?

The next proposition gives the affirmative answer to this question.

**Proposition 2.12.** If  $(\mathcal{M}, F)$  is generated by  $(M, A)$  then the action of  $P \in \text{End}_{q^e}(\mathcal{O}_X)$  on  $\mu_0(m)$  is given by  $P\mu_0(m) = \mu_e(P(F_X^{(e-1)*}(A) \dots Am))$  where the action of  $P$  on  $F_X^{e*} M$  is the natural one.

*Proof.* Set  $B = F_X^{(e-1)*}(A) \dots A$  and let  $\xi^{-1} : F_X^{e*} \mathcal{M} \rightarrow \mathcal{M}$  denote the  $e^{th}$ -structure morphism.  $P\mu_0(m)$  is defined as  $\xi^{-1} P\xi(\mu_0(m))$ . By definition,  $\mu_0(m) = \mu_e(Bm)$  and  $Bm = \sum_j f_j \otimes m_j \in F_X^{e*} M$ . Thus,  $\xi(\mu_0(m)) = \sum_j f_j \otimes \mu_0(m_j)$  because

$$\begin{aligned} \xi^{-1}(\sum_j f_j \otimes \mu_0(m_j)) &= \sum_j f_j F^e \mu_0(m_j) \\ &= \sum_j f_j \mu_e(1 \otimes m_j) \\ &= \mu_e(\sum_j f_j \otimes m_j) \\ &= \mu_e(Bm) \\ &= \mu_0(m). \end{aligned}$$

Hence,  $P\xi(\mu_0(m)) = \sum_j (Pf_j) \otimes \mu_0(m_j)$  and

$$\begin{aligned} P\mu_0(m) &= \xi^{-1} P\xi(\mu_0(m)) \\ &= \xi^{-1}(\sum_j (Pf_j) \otimes \mu_0(m_j)) \\ &= \sum_j (Pf_j) F^e(\mu_0(m_j)) \\ &= \sum_j (Pf_j) \mu_e(1 \otimes m_j) \\ &= \mu_e(\sum_j (Pf_j) \otimes m_j) \\ &= \mu_e(P(\sum_j f_j \otimes m_j)) \\ &= \mu_e(P(Bm)). \end{aligned}$$

□

The next corollary will be useful in understanding the action of the Euler operators in terms of the generating morphism.

**Corollary 2.13.** If  $q = p^\gamma$ ,  $X = X' \times_{\mathbb{k}} \text{Spec}(\mathbb{k}[t])$ ,  $M$  is a unit  $F_X$ -module, and  $\{\theta_i\}_{i=1}^{\gamma_e} \in \mathbb{D}_X^e$  the operators  $\theta_i = t^{p^{i-1}} \partial_t^{[p^{(i-1)}]}$  then

$$\mathbb{D}_R^e[t, \theta_1, \dots, \theta_{\gamma_e}] \mu_0(M) = \mu_e(\mathbb{D}_R^e[t, \theta_1, \dots, \theta_{\gamma_e}] F_X^{(e-1)*}(A) \dots AM)$$

and

$$\mathbb{D}_R^e[t, \theta_1, \dots, \theta_{\gamma_e}] \mu_0(tM) = \mu_e(\mathbb{D}_R^e[t, \theta_1, \dots, \theta_{\gamma_e}] t F_X^{(e-1)*}(A) \dots AM).$$

In particular, instead of studying the action of the Euler operators on

$$\mathbb{D}_R^e[t, \theta_1, \dots, \theta_{\gamma_e}]\mu_0(M)/\mathbb{D}_R^e[t, \theta_1, \dots, \theta_{\gamma_e}]\mu_0(tM)$$

it is enough to study their action on

$$\mathbb{D}_R^e[t, \theta_1, \dots, \theta_{\gamma_e}]F_X^{(e-1)*}(A)\dots AM/\mathbb{D}_R^e[t, \theta_1, \dots, \theta_{\gamma_e}]tF_X^{(e-1)*}(A)\dots AM.$$

## 2.4 $[\frac{1}{q^e}]$ powers of (non-unit) submodules

This section will study fractional powers of (non-unit) submodules of unit  $F$ -modules. This material generalizes the usual study of fractional powers of ideals.

**Definition 2.14.** If  $(\mathcal{M}, F)$  is a unit  $F$ -module and  $\mathcal{N} \subset \mathcal{M}$  is any subsheaf of  $\mathcal{O}_X$ -submodules (possibly not unit  $F$ ) then define  $\mathcal{N}^{[q^e]}$  to be the image under the restriction of the structure map  $F_X^{e*}\mathcal{N} \rightarrow \mathcal{M}$  of  $(\mathcal{M}, F' = F^e)$  (see 2.6 with  $\gamma = e$ ). We can also define fractional powers by defining  $\mathcal{N}^{[\frac{1}{q^e}]}$  to be the minimal module  $\mathcal{N}'$  with  $\mathcal{N}'^{[q^e]} \supset \mathcal{N}$ .

*Proof.* It is necessary to show that the latter definition is well-defined by showing a minimal such submodule exists. By taking  $\mathcal{N}' = \mathcal{M}$ , it follows that the set of submodules with this property is non-empty.  $F_X$  is flat and thus  $F_X^*$  naturally commutes with arbitrary intersections, thus  $\cap \mathcal{N}'_i^{[q^e]} = (\cap \mathcal{N}'_i)^{[q^e]}$ . The minimal element is the intersection of all elements from the set  $\{\mathcal{N}' | \mathcal{N}'^{[q^e]} \supset \mathcal{N}\}$ .  $\square$

**Example 2.15.** Consider the unit  $F$ -module  $(R, F_R)$  and let  $J, J' \subset R$  be ideals. The image of  $J'$  under the  $e^{th}$ -structure map  $F_R^{e*}J' \rightarrow R$  is precisely the ideal  $J'^{[q^e]} = (\{j^{q^e} | j \in J'\})$ . Hence, applying the definition of fractional powers given above to  $J$  results in the same ideal convention

The next proposition shows that these (fractional) powers are stable under étale pull-back.

**Proposition 2.16.** If  $(\mathcal{M}, F)$  is a unit  $F$ -module and  $\mathcal{N} \subset \mathcal{M}$  any subsheaf of  $\mathcal{O}_X$ -submodules (possibly not unit  $F$ ) and  $\pi : Y \rightarrow X$  is a flat morphism then

1.  $\pi^*(\mathcal{N}^{[q^e]}) = (\pi^*\mathcal{N})^{[q^e]}$ .
2.  $\pi^*(\mathcal{N}^{[\frac{1}{q^e}]}) = (\pi^*\mathcal{N})^{[\frac{1}{q^e}]}$  if  $\pi$  is étale.

*Proof.* By replacing  $q$  by  $q' = q^e$ , it is enough to prove these statements when  $e = 1$ .

1. The unit  $F$  structure on  $\pi^*(\mathcal{M})$ ,

$$\theta_Y^{-1} : F_Y^*(\pi^*\mathcal{M}) \rightarrow \pi^*\mathcal{M},$$

is defined (locally) by

$$f \otimes g \otimes m \mapsto fg^p \otimes F(m).$$

By definition,

$$\mathcal{N}^{[q]} = \text{Im}(\theta_X^{-1}|(F_X^* \mathcal{N}) \rightarrow \mathcal{M})$$

and

$$(\pi^* \mathcal{N})^{[q]} = \text{Im}(\theta_Y^{-1}|(F_Y^* \pi^* \mathcal{N}) \rightarrow \pi^* \mathcal{M})$$

There is a natural isomorphism  $\eta : \pi^* \circ F_X^* \cong F_Y^* \circ \pi^*$ . This isomorphism has the property that  $\theta_Y^{-1} \circ \eta = \pi^* \theta_X^{-1}$  and  $\eta(\pi^* F_X^* \mathcal{N}) = F_Y^* \pi^* \mathcal{N}$ . The right exactness of pull-back implies the image of  $\pi^* F_X^* \mathcal{N}$  under  $\pi^* \theta_X^{-1}$  is  $\pi^*(\mathcal{N}^{[q]})$ . Applying  $\eta$ , this image is also the image of  $F_Y^* \pi^* \mathcal{N}$  under  $\theta_Y^{-1}$  which is  $(\pi^* \mathcal{N})^{[q]}$ .

2. As étale maps are open, it is enough to prove this proposition only when  $\pi$  is the inclusion of a Zariski neighborhood or when  $\pi$  is a finite Galois cover.

Case 1:  $\pi = j : Y = U \rightarrow X$  is the inclusion of a Zariski open neighborhood.

The inclusion  $\mathcal{N}^{[\frac{1}{q}]}|_U \supset (\mathcal{N}|_U)^{[\frac{1}{q}]}$  follows from part 1. For the reverse inclusion, suppose  $\mathcal{N}' \subset \mathcal{M}|_U$  is such that  $\mathcal{N}'^{[q]} \supset \mathcal{N}|_U$ . Define  $\widetilde{\mathcal{N}}' = \text{Kern}(\mathcal{M} \rightarrow j_*(\mathcal{M}|_U/\mathcal{N}')) \subset \mathcal{M}$  then  $\widetilde{\mathcal{N}}'|_U = \mathcal{N}'$ .

By the flatness of Frobenius,

$$F_X^* \widetilde{\mathcal{N}}' = \text{Kern}(F_X^* \mathcal{M} \rightarrow F_X^* j_*(\mathcal{M}|_U/\mathcal{N}')).$$

We want to show that  $\widetilde{\mathcal{N}}'^{[q]} \supset \mathcal{N}$ . It is enough to show that image presheaf  $\theta_X^{-1}(F_X^* \widetilde{\mathcal{N}}')$  contains  $\mathcal{N}$ . Fix an open set  $V \subset X$  and a section  $n \in \mathcal{N}(V)$  then  $n|_{U \cap V}$  is zero in  $\mathcal{M}|_U/\mathcal{N}'^{[q]}(U \cap V) \cong \mathcal{M}|_U/\widetilde{\mathcal{N}}'|_U^{[q]}(U \cap V)$  where the isomorphism is the one given in part 1. We find  $\theta_X(V)(n)$  is in  $\text{Kern}(F_X^* \mathcal{M} \rightarrow F_X^* j_*(\mathcal{M}|_U/\mathcal{N}'))(V) = F_X^* \widetilde{\mathcal{N}}'(V)$  which yields  $n \in \widetilde{\mathcal{N}}'^{[q]}(V)$ . Thus  $\widetilde{\mathcal{N}}' \supset \mathcal{N}^{[\frac{1}{q}]}$  which implies by 1 that  $\mathcal{N}' \supset \mathcal{N}^{[\frac{1}{q}]|_U}$ . The result is obtained by taking  $\mathcal{N}' = (\mathcal{N}|_U)^{[\frac{1}{q}]}$ .

Case 2:  $\pi$  is a finite Galois cover.

Denote by  $G = \text{Aut}_X(Y)$  the Galois group. By part 1, it is clear  $\pi^*(\mathcal{N}^{[\frac{1}{q}]}) \supset (\pi^* \mathcal{N})^{[\frac{1}{q}]}$ . For the reverse implication, suppose that  $\mathcal{N}'^{[q]} \supset \pi^* \mathcal{N}$ . As  $\pi^* \mathcal{N}$  is closed under the natural  $G$ -action on  $\pi^* \mathcal{M}$ ,  $(g\mathcal{N}')^{[q]} \supset \pi^* \mathcal{N}$  for all  $g \in G$ . Therefore  $(\pi^* \mathcal{N})^{[\frac{1}{q}]}$  is a  $G$ -submodule of  $\pi^* \mathcal{M}$ . If  $\mathcal{N}'' = (\pi^* \mathcal{N})^{[\frac{1}{q}]}^G \subset \mathcal{M}$  then by Galois descent,  $(\pi^* \mathcal{N})^{[\frac{1}{q}]} = \pi^*(\mathcal{N}'')$ . In particular by part 1,  $\pi^*(\mathcal{N}''^{[q]}) = (\pi^* \mathcal{N}'')^{[q]} \supset \pi^* \mathcal{N}$ . As  $\pi^*$  is faithfully

flat, it must be the case that  $\mathcal{N}''^{[q]} \supset \mathcal{N}$ . Hence,  $\mathcal{N}'' \supset \mathcal{N}^{[\frac{1}{q}]}$  and  $\pi^* \mathcal{N}'' = (\pi^* \mathcal{N})^{[\frac{1}{q}]} \supset \pi^* (\mathcal{N}^{[\frac{1}{q}]})$ .

□

**Corollary 2.17.** If  $(\mathcal{M}, F)$  is a unit  $F$ -module on  $X$ ,  $\mathcal{N} \subset \mathcal{M}$  a (possibly not unit  $F$ )  $\mathcal{O}_X$ -submodule, and  $x \in X$  then formation of  $[q^e]$  and  $[\frac{1}{q^e}]$  powers commutes with restriction to étale neighborhoods of  $x$ .

There is also the following generalization of [BMS08, 2.5].

**Theorem 2.18.** Let  $l$  be an integer,  $(R^{\oplus l}, F_R^{\oplus l})$  the natural unit  $F$ -module on  $R$  of rank  $l$ , and  $N \subset R^{\oplus l}$  a submodule generated by  $v_1, \dots, v_n$ . If  $F_{R*}^e R$  is a free  $R$ -module with  $R$ -basis  $b_1, \dots, b_c$  ( $c = q^{e \dim(R)}$ ),  $b_{1,1}, \dots, b_{1,l}, \dots, b_{c,l}$  the corresponding basis for  $F_{R*}^e R^{\oplus l}$  and  $v_i = \sum_{j=0, j'=0}^{c,l} a_{i,j,j'}^{q^e} b_{j,j'}$  then setting  $w_{i,j} = \langle a_{i,j,1}, \dots, a_{i,j,l} \rangle$  we have

$$N^{[\frac{1}{q^e}]} = \sum_{i,j} R w_{i,j}.$$

*Proof.* The proof follows by a direct adaptation of the proof of [BMS08, 2.5]. To ease notation, we again note it is enough to prove the case  $e = 1$ .

The inclusion  $N^{[\frac{1}{q}]} \subset (\{w_{i,j}\})$  is obvious because  $v_i = \sum_j b_j F_R(w_{i,j})$ .

The reverse inclusion is a consequence of the following observation. If  $\{b_j^\vee\}$  is the dual basis of  $F_{R*} R$  over  $R$ , then for any  $N'$ ,  $N'^{[q]}$  is closed under application of  $b_j^\vee$  (acting along the diagonal). In particular, if  $N'^{[q]} \supset N$  then  $b_j^\vee(v_i) = F_R(w_{i,j}) \in N'^{[q]}$ . As  $F_R$  is faithfully flat,  $w_{i,j} \in N'$ .

□

**Corollary 2.19.** Let  $R$  be a polynomial ring,  $l$  an integer, and  $(R^{\oplus l}, F_R^{\oplus l})$  the natural unit  $F$ -module on  $R$  of rank  $l$ . If  $N \subset R^{\oplus l}$  can be generated by elements of degree at most  $d$ , where the degree of a vector is defined to be the maximum of the degree of each entry, then  $N^{[\frac{1}{q^e}]}$  can be generated by elements of degree at most  $\lfloor \frac{d}{q^e} \rfloor$ .

*Proof.* Follows directly from 2.18.

□

## 2.5 $\mathbb{D}_X^e$ -submodules of unit $F$ -modules

This section will briefly generalize [BMS09, 2.2] from ideals to arbitrary unit  $F^e$ -modules.

**Theorem 2.20.** If  $(\mathcal{M}, F)$  is a unit  $F$ -module then the  $\mathbb{D}_X^e$ -submodules of  $\mathcal{M}$  are precisely those of the form  $\mathcal{N}^{[q^e]}$  for  $\mathcal{N}$  a (possibly not unit  $F$ )  $\mathcal{O}_X$ -submodule of  $\mathcal{M}$ . In particular, the  $\mathbb{D}_X^e$ -module generated by  $\mathcal{N}$  is  $\mathcal{N}^{[\frac{1}{q^e}]^{[q^e]}}$ .

*Proof.* The  $\mathbb{D}_R^e$ -structure of  $\mathcal{M}$  is given by the  $\mathbb{D}_R^e$ -structure of  $F^{e*}\mathcal{M}$  after identification through the  $e^{th}$  structure isomorphism. By Morita equivalence, the  $F^{e*}\mathcal{M}$ -submodules of  $\mathcal{M}$  are precisely those of the form  $F^{e*}\mathcal{N}$  for  $\mathcal{N} \subset \mathcal{M}$  an  $\mathcal{O}_X$ -submodule. Applying the  $e^{th}$  structure isomorphism to  $\mathcal{N}$  gives the result. Alternatively, one can modify the proof of [BMS09, 2.2] to obtain this result by direct computation.  $\square$

### 3 List Test Modules

We will now discuss a generalization of the test ideals  $\tau(f^\alpha)$  which will be used in the next section to prove the existence of  $b$ -functions. A good review of test ideals and  $F$ -jumping exponents is contained in [BMS08]. The most relevant result of [BMS08] to be generalized in this section is that  $F$ -jumping exponents are discrete and rational. Unlike test ideals, which are defined by two parameters  $\lambda$  and  $e$  with  $\lambda$  fixed and  $e \gg 0$ , list test modules will exhibit their most interesting behavior by fixing  $e$  and allowing  $\lambda$  to vary. This technical variation requires the analogous definition of  $F$ -jumping exponents to be more analytic in nature. Example 3.4 will detail the relationship between list test modules and generalized test ideals  $\tau(f^\alpha)$  and includes an analysis of the relationship between the jumping numbers and  $F$ -jumping exponents. The main result of this section is that the jumping numbers associated to list test modules are discrete and rational.

List test modules will be determined by a list of matrices. In order to assist the reader, in the first subsection a thorough investigation of list test modules is done when the matrices in the list are  $1 \times 1$  and the list is of length less than  $q$ . In this case, they will actually determine an ideal in  $R$  and hence are called “simple list test ideals”. The generalized test ideals  $\tau(f^\alpha)$  are examples of simple list test ideals. The second subsection contains the generalization of the important statements from simple list test ideals to the list test modules. The proofs of the general case are omitted because they follow by the analogous arguments.

#### 3.1 Simple list test ideals

To provide appropriate motivation for the definition of list test modules and their jumping numbers, we will first analyze a special case and its relation to test ideals and  $F$ -jumping exponents.

**Notation 3.1.** We make a small modification of classical notion. Let  $S = \cup_e S_e$  a union of finite sets indexed by  $\mathbb{N}$ . We say that  $s$  is an accumulation point of  $S$  if it is an element of  $\cap_{e'} \overline{\cup_{e \geq e'} S_e}$ . That is, the accumulation points of  $S$  are the usual accumulation points but we also allow limit points of constant sequences as the constant is contained in  $S_e$  for infinitely many  $e$ .

**Definition 3.2.** Let  $r_0, \dots, r_{q-1} \in R$  be a list of length  $q$ . For each  $\lambda \in (0, 1]$  and  $e \geq 0$  define,

$$I(r_0, \dots, r_{q-1}, \lambda, e) = (r_{i_0} r_{i_1}^q \dots r_{i_e}^{q^e})^{\lfloor \frac{1}{q^{e+1}} \rfloor}$$

where  $\lceil \lambda q^{e+1} \rceil - 1 = i_0 + i_1 q + \dots + i_e q^e$  is the unique base  $q$  expansion of  $\lceil \lambda q^{e+1} \rceil - 1$ .

The  $e^{\text{th}}$  **simple list test ideal** is,

$$\tau(r_0, \dots, r_{q-1}, \lambda, e) = \sum_{\lambda' \leq \lambda} I(r_0, \dots, r_{q-1}, \lambda', e).$$

This definition can be extended to all  $\lambda \in \mathbb{R}$  by setting

$$\tau(r_0, \dots, r_{q-1}, \lambda, e) = \tau(r_0, \dots, r_{q-1}, \lambda_0, e)$$

where  $\lambda_0$  is the unique representative in  $(0, 1]$  of the class  $\bar{\lambda} \in \mathbb{R}/\mathbb{Z}$ .

**Remark 3.3.** For fixed  $\lambda$ , the ideals  $I(r_0, \dots, r_{q-1}, \lambda, e)$  decrease as  $e$  increases. This implies that for fixed  $\lambda$ , the simple list test ideals also decrease as  $e$  increases. For fixed  $e$ , the simple list test ideals are constructed to increase as  $\lambda$  increases. There is also lower semi-continuity; for every  $e$  and every  $\lambda$  there exists  $\lambda' < \lambda$  such that  $\tau(r_0, \dots, r_{q-1}, \lambda', e) = \tau(r_0, \dots, r_{q-1}, \lambda, e)$ .

The following example will demonstrate that the generalized test ideals  $\tau(f^\alpha)$  for  $f \in R$  are a special case of list test modules. It will also discuss how to recover  $F$ -jumping exponents as an analytic property of list test modules.

**Example 3.4.** Let  $R$  be a regular Noetherian  $F$ -finite ring, possibly not of essentially finite type over  $\mathbb{k}$ . If  $f \in R$  and we define  $r_k = f^{q^{-1}-k}$  then for all  $\lambda \in (0, 1]$

$$I(r_0, \dots, r_{q-1}, \lambda, e) = (r_{i_0} r_{i_1}^q \dots r_{i_e}^{q^e})^{\lfloor \frac{1}{q^{e+1}} \rfloor} = (f^{\sum_a (q-1-i_a)q^a})^{\lfloor \frac{1}{q^{e+1}} \rfloor} = (f^{q^{e+1} - \lceil \lambda q^{e+1} \rceil})^{\lfloor \frac{1}{q^{e+1}} \rfloor}$$

In particular, if  $\lambda' \leq \lambda$  then  $q^{e+1} - \lceil \lambda' q^{e+1} \rceil \geq q^{e+1} - \lceil \lambda q^{e+1} \rceil$ ,

$$I(r_0, \dots, r_{q-1}, \lambda', e) \subset I(r_0, \dots, r_{q-1}, \lambda, e),$$

and

$$I(r_0, \dots, r_{q-1}, \lambda, e) = \tau(r_0, \dots, r_{q-1}, \lambda, e).$$

As  $R$  is Noetherian, for any  $\alpha \in (0, 1)$  there exists  $e \gg 0$  such that  $\tau(f^\alpha) = (f^{\lceil \alpha q^{e+1} \rceil})^{\lfloor \frac{1}{q^{e+1}} \rfloor}$ . If  $\lceil \alpha q^{e+1} \rceil - 1 = i_0 + i_1 q + \dots + i_e q^e$  is the base  $q$ -expansion and  $e \gg 0$ , setting  $\lambda = \lambda_{\alpha, e} = \sum_{k=0}^e (q-1-i_k)q^{k-e-1}$  implies  $\lambda \in (0, 1]$ ,  $q^{e+1} - \lceil \lambda q^{e+1} \rceil = \lceil \alpha q^{e+1} \rceil$ , and

$$\tau(r_0, \dots, r_{q-1}, \lambda, e) = (f^{q^{e+1} - \lceil \lambda q^{e+1} \rceil})^{\lfloor \frac{1}{q^{e+1}} \rfloor} = \tau(f^\alpha).$$

If  $R$  is of essentially finite type over  $\mathbb{k}$ , then the  $F$ -jumping exponents are discrete and one may choose a single  $e$  large enough to simultaneously obtain all test ideals  $\tau(f^\alpha)$  in this fashion.

Recall the definition of an  $F$ -jumping exponent from [BMS08, 2.17].

$\alpha \in (0, 1)$  is called an  $F$ -jumping exponent if for all  $\alpha' < \alpha$ ,

$$\tau(f^{\alpha'}) \neq \tau(f^\alpha).$$

Suppose  $\alpha$  is an  $F$ -jumping exponent. For each  $n$ , there is a number  $\alpha'$  with  $\alpha - \frac{1}{n} < \alpha' < \alpha$  and  $\tau(f^\alpha) \neq \tau(f^{\alpha'})$ . Inductively, we may choose  $e_n \geq e_{n-1} \gg 0$  such that,

$$\tau(f^\alpha) = (f^{\lceil \alpha q^{e_n+1} \rceil})^{\lfloor \frac{1}{q^{e_n+1}} \rfloor} \neq (f^{\lceil \alpha' q^{e_n+1} \rceil})^{\lfloor \frac{1}{q^{e_n+1}} \rfloor} = \tau(f^{\alpha'}).$$

In particular, there exists an integer  $m_n$  with

$$\lceil \alpha q^{e_n+1} \rceil \geq m_n > \lceil (\alpha - \frac{1}{n}) q^{e_n+1} \rceil$$

such that

$$(f^{m_n})^{\lfloor \frac{1}{q^{e_n+1}} \rfloor} \neq (f^{m_n-1})^{\lfloor \frac{1}{q^{e_n+1}} \rfloor}.$$

Consider the sequence  $\alpha_n = \frac{m_n}{q^{e_n+1}}$ . It is clear from the construction that  $\frac{m_n}{q^{e_n+1}}$  and  $\frac{m_n-1}{q^{e_n+1}}$  converge to  $\alpha$ . By the definitions given above, the sequence  $\lambda_{\alpha_n, e_n}$  converges to  $1 - \alpha$ . We wish to make an alternative definition which recovers the set of jumping numbers in  $(0, 1)$  but in the notation of list test ideals.

Define

$$S_e = \{\lambda \in (0, 1) \mid \tau(r_0, \dots, r_{q-1}, \lambda, e) \neq \tau(r_0, \dots, r_{q-1}, \lambda', e) \forall 1 \geq \lambda' > \lambda\}.$$

Notice that elements of  $S_e$  are all of the form  $\frac{a}{q^{e+1}}$  for  $0 < a < q^{e+1}$ . Moreover, each  $\lambda_{\alpha_n, e_n} \in S_{e_n}$  as  $\alpha_n$  were chosen such that

$$(f^{m_n})^{\lfloor \frac{1}{q^{e_n+1}} \rfloor} = \tau(r_0, \dots, r_{q-1}, \lambda_{\alpha_n, e_n}, e_n) \neq \tau(r_0, \dots, r_{q-1}, \lambda_{\alpha_n, e_n} + \frac{1}{q^{e_n+1}}, e_n) = (f^{m_n-1})^{\lfloor \frac{1}{q^{e_n+1}} \rfloor}.$$

If we define  $S = \cup_e S_e$  then we have shown that the  $F$ -jumping exponents in  $(0, 1)$  are of the form  $1 - s$  for  $s$  an accumulation points of  $S$ .

It will now be shown that if  $s$  is an accumulation point of  $S$  then  $1 - s$  is an  $F$ -jumping exponent. Suppose  $s$  is an accumulation point and set  $\alpha = 1 - s$  then  $s$  is the limit of a sequence  $\lambda_n \in S_{e_n}$  and the sequence  $e_n$  is unbounded. Choose integers  $m_n$  so that  $\lambda_{\frac{m_n}{q^{e_n+1}}, e_n} = \lambda_n$ . The sequence  $\frac{m_n}{q^{e_n+1}}$  converges  $\alpha$ . We finish the analysis by breaking into cases.



Case 1:  $\frac{m_n}{q^{e_n+1}} \geq \alpha$  for all  $n \gg 0$ .

For  $n$  very large  $\frac{m_n}{q^{e_n+1}}$  nearly approximates  $\alpha$ . By [BMS08, 2.14],

$$\tau(f^\alpha) = (f^{m_n})^{\lfloor \frac{1}{q^{e_n+1}} \rfloor} \neq (f^{m_n-1})^{\lfloor \frac{1}{q^{e_n+1}} \rfloor} = \tau(f^{\frac{m_n-1}{q^{e_n+1}}}).$$

We have  $\frac{m_n-1}{q^{e_n+1}} < \alpha$ , for if not,  $\frac{m_n}{q^{e_n+1}}$  would closely approximate  $\alpha$  also and  $\tau(f^\alpha) = \tau(f^{\frac{m_n-1}{q^{e_n+1}}})$ . The sequence  $\frac{m_n-1}{q^{e_n+1}}$  converges to  $\alpha$  which implies that  $\alpha$  is an  $F$ -jumping exponent.

Case 2:  $\frac{m_n}{q^{e_n+1}} < \alpha$  infinitely often.

We may pass to a subsequence and assume for all  $n$  the inequality holds.

Fix  $\alpha' < \alpha$  then chosen  $n$  large enough that

$$\tau(f^\alpha) = (f^{\lceil \alpha q^{e_n+1} \rceil})^{\lfloor \frac{1}{q^{e_n+1}} \rfloor}, \tau(f^{\alpha'}) = (f^{\lceil \alpha' q^{e_n+1} \rceil})^{\lfloor \frac{1}{q^{e_n+1}} \rfloor},$$

and  $\alpha' < \frac{m_n-1}{q^{e_n+1}}$ .

Direct computation yields,

$$\tau(f^{\alpha'}) \supset (f^{m_n-1})^{\lfloor \frac{1}{q^{e_n+1}} \rfloor} \supset (f^{m_n})^{\lfloor \frac{1}{q^{e_n+1}} \rfloor} \supset \tau(f^\alpha)$$

where the middle containment is strict. Thus for any  $\alpha' < \alpha$ ,  $\tau(f^\alpha) \neq \tau(f^{\alpha'})$  so  $\alpha$  is an  $F$ -jumping exponent.

We will see in the proofs of 3.12 and 3.14 that the second case is actually null if  $R$  is of essentially finite type over  $\mathbb{k}$ . In this situation, the proof of the first case shows for very large  $e$ , elements of  $S_e$  are always less than  $\frac{1}{q^{e+1}}$  away from a jumping number.

Motivated by the alternate description of the  $F$ -jumping exponents in the previous example the following definition for jumping numbers is proposed.

**Definition 3.5.** For each  $e$ , set

$$S_e = \{\lambda \in (0, 1) \mid \tau(r_0, \dots, r_{q-1}, \lambda', e) \neq \tau(r_0, \dots, r_{q-1}, \lambda, e) \forall 1 \geq \lambda' > \lambda\}.$$

It is clear that the set  $S_e$  is contained in the set  $\{\frac{m}{q^{e+1}} \mid 0 < m < q^{e+1}\}$ .

Set  $S = \cup_e S_e$ . We define a **jumping number** for the list  $r_0, \dots, r_{q-1}$  to be a non-zero accumulation point of  $S$ . In the extended setting, we say  $\lambda$  is a jumping number if and only if  $\lambda$  is the integer translate of a jumping number.

**Remark 3.6.** It will be shown in the proofs of 3.12 and 3.14 that when  $R$  is of essentially finite type over  $\mathbb{k}$ , 0 is never an accumulation point of  $S$ . As this is the only case that is of interest to this article, it is most convenient to exclude 0 from the general definition.

The following proposition was proven in 3.4.

**Proposition 3.7.** If  $f \in R$  is a function then  $\lambda \in (0, 1)$  is a jumping number for the list  $r_k = f^{q^{-1}-k}$  if and only if  $1 - \lambda$  is a  $F$ -jumping exponent for  $f$ .

**Proposition 3.8.** If  $\lambda = \frac{m-1}{q^{e+1}} \in S_e$  and  $m-1$  is not divisible by  $q^e$  then the fractional part of  $\frac{m-1}{q^e}$  is in  $S_{e-1}$ .

*Proof.* Proceed by proving the contrapositive: If the fractional part of  $\frac{m-1}{q^e}$  is not in  $S_{e-1}$  then  $\frac{m-1}{q^{e+1}}$  is not in  $S_e$ .

Write  $m-1 = i_0 + i_1q + \dots + i_eq^e$  and suppose,

$$\tau(r_0, \dots, r_{q-1}, \lambda_0, e-1) = \tau(r_0, \dots, r_{q-1}, \lambda_0 + q^{-e}, e-1)$$

where  $\lambda_0 = \frac{i_0 + i_1q + \dots + i_{e-1}q^{e-1}}{q^e}$  is the fractional part of  $\frac{m-1}{q^e}$ .

The goal is to show that

$$\tau(r_0, \dots, r_{q-1}, \frac{m}{q^{e+1}}, e) \subset \tau(r_0, \dots, r_{q-1}, \frac{m-1}{q^{e+1}}, e).$$

It is enough to show that

$$(r_{i_0}r_{i_1}^q \dots r_{i_e}^{q^e})^{[\frac{1}{q^{e+1}}]} \subset \tau(r_0, \dots, r_{q-1}, \frac{m-1}{q^{e+1}}, e).$$

By the equality  $\mathbb{D}_R^e I = (I^{[\frac{1}{q^e}]})^{[q^e]}$  from 2.20 for any ideal  $I \subset R$ ,

$$\tau(r_0, \dots, r_{q-1}, \lambda_0, e-1) = \tau(r_0, \dots, r_{q-1}, \lambda_0 + q^{-e}, e-1)$$

implies

$$\begin{aligned} \tau(r_0, \dots, r_{q-1}, \lambda_0 + q^{-e}, e-1)^{[q^e]} &= \tau(r_0, \dots, r_{q-1}, \lambda_0, e-1)^{[q^e]} \\ &\cup \qquad \qquad \qquad \parallel \\ \mathbb{D}_R^e r_{i_0} r_{i_1}^q \dots r_{i_{e-1}}^{q^{e-1}} &\qquad \qquad \sum_{\lambda' \leq \lambda_0} \mathbb{D}_R^e r_{j_0} r_{j_1}^q \dots r_{j_{e-1}}^{q^{e-1}}. \end{aligned}$$

$\mathbb{D}_R^e$  is defined as the operators which commute with  $q^e$  powers, so

$$\mathbb{D}_R^e r_{i_0} r_{i_1}^q \dots r_{i_{e-1}}^{q^{e-1}} r_{i_e}^{q^e} \subset \sum_{\lambda' \leq \lambda_0} \mathbb{D}_R^e r_{j_0} r_{j_1}^q \dots r_{j_{e-1}}^{q^{e-1}} r_{i_e}^{q^e}.$$

This induces the containment

$$\mathbb{D}_R^{e+1} r_{i_0} r_{i_1}^q \dots r_{i_{e-1}}^{q^{e-1}} r_{i_e}^{q^e} \subset \sum_{\lambda' \leq \lambda_0} \mathbb{D}_R^{e+1} r_{j_0} r_{j_1}^q \dots r_{j_{e-1}}^{q^{e-1}} r_{i_e}^{q^e} \subset \sum_{\lambda' \leq \lambda} \mathbb{D}_R^{e+1} r_{j_0} r_{j_1}^q \dots r_{j_{e-1}}^{q^{e-1}} r_{j_e}^{q^e}$$

which implies, by a second application of 2.20, that

$$(r_{i_0}^q r_{i_1}^q \dots r_{i_e}^q)^{[\frac{1}{q^{e+1}}]^{[q^{e+1}]}} \subset \tau(r_0, \dots, r_{q-1}, \frac{m-1}{q^{e+1}}, e)^{[q^{e+1}]}$$

The claim is then proven by the faithful flatness of Frobenius. □

**Corollary 3.9.**  $\lambda \in (0, 1]$  is a jumping number for  $r_0, \dots, r_q$  then either the fractional part of  $q\lambda$  is a jumping number or  $\lambda = \frac{a}{q}$  with  $0 < a \leq q$ .

*Proof.* We prove that if  $\lambda$  is not of the form  $\frac{a}{q}$  for  $0 < a \leq q$  then the fractional part of  $q\lambda$  is a jumping number.

Claim: There exists a sequence  $\frac{m_n}{q^{e_n+1}} \rightarrow \lambda$  with

1.  $\frac{m_n}{q^{e_n+1}} \in S_{e_n}$ .
2.  $e_n \rightarrow \infty$ .
3.  $m_n$  is not divisible by  $q^{e_n}$ .

We prove the claim by contradiction. Assume no such sequence exists. As  $\lambda$  is an accumulation point, we know there is a sequence  $\frac{m_n}{q^{e_n+1}}$  converging to  $\lambda$  with  $\frac{m_n}{q^{e_n+1}} \in S_{e_n}$  and  $e_n \rightarrow \infty$ . By the contradiction assumption, after dropping finitely many terms, we may assume  $m_n$  is always divisible by  $q^{e_n}$ . Write  $m_n = a_n q^{e_n}$  with  $0 < a_n < q$  an integer. Dividing both sides by  $q^{e_n}$  and taking the limit shows that

$$\lambda q = \lim_n a_n.$$

The integers  $a_n$  therefore must eventually be the constant  $a = \lambda q$ . In particular,  $\lambda = \frac{a}{q}$ , a contradiction.

Let  $\frac{m_n}{q^{e_n+1}}$  be a sequence as in the claim. By considering a subsequence, we may assume that for all  $n$   $m_n$  is not divisible by  $q^{e_n}$ . By the previous proposition, the fractional parts of  $\frac{m_n}{q^{e_n}}$  are in  $S_{e_n-1}$  for all  $n$ . The sequence  $\frac{m_n}{q^{e_n}}$  converging to  $q\lambda$  in  $\mathbb{R}$  implies the sequence  $\frac{m_n}{q^{e_n}}$  converges in  $\mathbb{R}/\mathbb{Z}$  to  $\overline{q\lambda}$ . As the fractional part of  $q\lambda \neq 0$ , the fractional part of  $q\lambda$  is the unique  $(0, 1]$  representative of  $\overline{q\lambda} \in \mathbb{R}/\mathbb{Z}$ . Therefore, the sequence of fractional parts of  $\frac{m_n}{q^{e_n}}$  converges to the fractional part of  $q\lambda$  and hence the fractional part of  $q\lambda$  is an accumulation point of  $S$ . □

**Theorem 3.10.** If  $\lambda$  is a jumping number in the extended sense then either  $q\lambda$  is a jumping number or  $q\lambda$  is an integer. In particular, if  $q\lambda$  is not an integer then the fractional part of  $q\lambda$  is a jumping number.

*Proof.* We proceed by proving if  $q\lambda$  is not an integer then  $q\lambda$  is a jumping number. By assumption, there exists an integer  $j$  such that  $\frac{m_n}{q^{e_n+1}} + j \rightarrow \lambda$  where  $\frac{m_n}{q^{e_n+1}} \in S_{e_n}$ . As  $q\lambda$  is not an integer, we know that  $\lambda - j \neq \frac{a}{q}$  with  $0 < a \leq q$ .

By 3.9, the fractional part of  $q\lambda - qj$  is a jumping number. Hence  $q\lambda - qj$  is a jumping number (in the extended sense) which implies  $q\lambda$  is a jumping number.  $\square$

The next proposition will give us insight into the behavior of list test ideals under the formation of quotient rings.

**Proposition 3.11.** Let  $x_1, \dots, x_n \in R$  be a regular sequence defining a maximal ideal  $m \subset R$  such that

1.  $F_{R*}^{(e+1)}R$  is freely generated over  $R$  by  $x^{\vec{u}}$  (in multi-index notation) for  $0 \leq \vec{u} \leq q^{e+1} - 1$ .
2.  $F_{S*}^{(e+1)}S$  is freely generated over  $S$  by  $\widetilde{x^{\vec{v}}}$  where  $Q : R \rightarrow S = R/(x_n)$ .

If  $r_0, \dots, r_{q-1}$  is a list in  $S$  and  $\widetilde{r_0}, \dots, \widetilde{r_{q-1}}$  are representatives in  $R$  with  $Proj_{x^{\vec{u}}}(\widetilde{r_k}) = 0$  for all  $\vec{u}$  with non-zero  $n^{th}$  component then

$$I(\widetilde{r_0}, \dots, \widetilde{r_{q-1}}, \lambda, e) + (x_n) = Q^{-1}(I(r_0, \dots, r_{q-1}, \lambda, e))$$

and

$$\tau(\widetilde{r_0}, \dots, \widetilde{r_{q-1}}, \lambda, e) + (x_n) = Q^{-1}(\tau(r_0, \dots, r_{q-1}, \lambda, e)).$$

*Proof.* Write  $\lceil \lambda q^{e+1} \rceil - 1 = i_0 + i_1 q + \dots + i_e q^e$  and set

$$\widetilde{r} = \widetilde{r_{i_0}} \dots \widetilde{r_{i_e}}^{q^e}.$$

We want to show that

$$Q^{-1}((r)^{\lceil \frac{1}{q^{e+1}} \rceil}) = (\widetilde{r})^{\lceil \frac{1}{q^{e+1}} \rceil} + (x).$$

Reindex the set  $\{x^{\vec{v}} \mid \text{The } n^{th} \text{ component of } \vec{v} \text{ is } 0\}$  by  $\{b_i\}_{i=0}^{q^{(e+1)(n-1)}-1}$ . We have that  $\overline{b_i}$  is a basis for  $F_{S*}S$  over  $S$  and  $b_i x_n^j$  for  $0 \leq i < q^{(e+1)(n-1)}$  and  $0 \leq j < q$  is a basis for  $F_{R*}R$  over  $R$ .

Write  $\widetilde{r} = \sum_{i,j} a_{ij}^{q^{e+1}} b_i x_n^j$  then, by the assumption  $Proj_{x^{\vec{u}}}(\widetilde{r_k}) = 0$  for all  $\vec{u}$  with non-zero  $n^{th}$  component,  $a_{ij} = 0$  for all  $j > 0$ . By 2.18

$$I(r_0, \dots, r_{q-1}, \lambda, e) = (\{\overline{a_{i0}}\})$$

and

$$I(\widetilde{r_0}, \dots, \widetilde{r_{q-1}}, \lambda, e) = (\{a_{ij}\}) = (\{a_{i0}\}).$$

Thus,

$$Q^{-1}(I(r_0, \dots, r_{q-1}, \lambda, e)) = (\{a_{i0} \mid 0 \leq i < q^{e(n-1)-1}\} \cup \{x_n\}) = I(\widetilde{r_0}, \dots, \widetilde{r_{q-1}}, \lambda, e) + (x_n).$$

The analogous result for simple list test ideals follows immediately.  $\square$

**Lemma 3.12.** If  $R$  is a polynomial ring then the jumping numbers for  $r_0, \dots, r_{q-1}$  in  $(0, 1]$  are finite and rational.

*Proof.* Let  $d$  be the maximum of the degrees of  $\{r_i\}$ . It follows from 2.19 that  $I(r_0, \dots, r_{q-1}, \lambda, e)$  can be generated by elements with degree less than or equal to  $\frac{d(q^{e+1}-1)}{(q-1)q^{e+1}}$  for any  $\lambda$  or  $e$ . These numbers increase with  $e$  and hence  $I(r_0, \dots, r_{q-1}, \lambda, e)$  can be generated by elements with degree less than or equal to  $\lfloor \frac{d}{q-1} \rfloor = \lfloor \lim_e \frac{d(q^{e+1}-1)}{(q-1)q^{e+1}} \rfloor$ . By construction, the same is true for the ideals  $\tau(r_0, \dots, r_{q-1}, \lambda, e)$ .

If  $W$  is the vector space of polynomials of degree less than or equal to  $\lfloor \frac{d}{q-1} \rfloor$ , then

$$\tau(r_0, \dots, r_{q-1}, \lambda', e) = \tau(r_0, \dots, r_{q-1}, \lambda, e)$$

if and only if they are equal after intersection with  $W$ . If we set  $W_{\lambda,e} = \tau(r_0, \dots, r_{q-1}, \lambda, e)$ , then for fixed  $e$  the collection  $\{W_{\lambda,e}\}$  is a sequence of vector subspaces of  $W$ . Therefore the cardinalities of sets  $S_e$  are uniformly bounded by the vector space length of  $W$  plus one.

Consider the collection of finite sets  $T_e = \{0\} \cup S_e$ . This collection of sets has the property that if  $t$  is an accumulation point of  $\cup_e T_e$  then the fractional part of  $qt$  is also an accumulation point by 3.10. Moreover, we will now prove that the set  $\cup_e T_e$  has no strictly decreasing sequences of length more than  $\text{length}(W) + 2$  by showing that  $\cup_e S_e$  has no strictly decreasing lists of length more than  $\text{length}(W) + 1$ .

Let  $s_n \in S_{e_n}$  be a list of length  $N'$  with  $s_{n+1} < s_n$  and  $e_{n+1} \geq e_n$ . We have

$$\tau(r_0, \dots, r_{q-1}, s_{n+1}, e_{n+1}) \subset \tau(r_0, \dots, r_{q-1}, s_n, e_{n+1}) \subset \tau(r_0, \dots, r_{q-1}, s_n, e_n)$$

with the first containment strict as  $s_{n+1} \in S_{e_{n+1}}$ . This gives us the strictly decreasing sequence with  $N'$  terms,

$$\dots \subset \tau(r_0, \dots, r_{q-1}, s_{n+1}, e_{n+1}) \subset \tau(r_0, \dots, r_{q-1}, s_n, e_n) \subset \dots \subset \tau(r_0, \dots, r_{q-1}, s_1, e_1)$$

Repeating the argument as before, these containments are strict if and only if they are strict after intersection with  $W$ . This implies  $N' \leq \text{length}(W) + 1$ .

The sets  $T_e$  satisfy the axioms of the next proposition and thus the set of its accumulation points are finite and rational.  $\square$

**Proposition 3.13.** If  $T_e \subset [0, 1]$  is a collection of subsets with the properties:

1. There exists  $N$  such that  $\cup_e T_e$  contains no length  $N$  lists  $t_n \in T_{e_n}$  with  $t_{n+1} < t_n$  and  $e_{n+1} \geq e_n$ .
2. If  $\lambda$  is an accumulation point then so is the fractional part of  $q\lambda$ .

then the set of accumulation points of  $T_e$  is a finite set of rational numbers.

*Proof.* We start by proving the finiteness statement by contradiction. Suppose there are distinct  $N + 1$  accumulation points labeled in decreasing order,  $\lambda_1 > \lambda_2 > \dots > \lambda_{N+1}$ . Choose  $\epsilon$  small enough so that each accumulation point is at least  $3\epsilon$  away from the other accumulation points. Choose  $e_1$  and  $t_1 \in S_{e_1}$  such that  $|\lambda - t_1| < \epsilon$ . Inductively, choose  $e_{i+1} \geq e_i$  and  $t_{i+1} \in S_{e_{i+1}}$  with  $|\lambda_{i+1} - t_{i+1}| < \epsilon$ . Notice that

$$t_i - t_{i+1} \geq -|t_i - \lambda_i| + (\lambda_i - \lambda_{i+1}) - |t_{i+1} - \lambda_{i+1}| \geq -\epsilon + 3\epsilon - \epsilon = \epsilon.$$

Thus, the  $t_i$  form a strictly decreasing sequence of length  $N + 1$ . A contradiction to the assumption that our sets satisfy condition 1.

To prove rationality, consider the set of fractional parts of  $q^k \lambda$  as  $k$  varies. These are all accumulation points by condition 2. We have already shown that the number of accumulation points is finite so there must exist  $n$  and  $m$  such that the fractional parts of  $q^n \lambda$  and  $q^m \lambda$  are equal. That is,  $(q^n - q^m)\lambda$  is an integer  $t$ . This implies  $\lambda$  is the rational number  $\frac{t}{q^n - q^m}$ .  $\square$

Similar to the proof of the discreteness and rationality of  $F$ -jumping exponents in [BMS08], one can deduce the general case for  $R$  a regular  $F$ -finite ring of essentially finite type over  $\mathbb{k}$ , from the case of the polynomial ring. The strategy of the proof is similar to the strategy for the proof in [BMS08]. First, we use the smoothness of  $R$  and 2.17 to reduce to the case of a standard étale neighborhood of a polynomial ring. The result then follows from the case of a polynomial ring by using 3.11.

**Theorem 3.14.** If  $R$  is smooth and of essentially finite type over  $\mathbb{k}$  then the (extended) jumping numbers for the list  $r_0, \dots, r_{q-1}$  are discrete and rational.

*Proof.* As  $R$  is smooth and of essentially finite type over  $\mathbb{k}$ , we may cover  $X = \text{Spec}(R)$  by a finite collection of Zariski-open subsets  $U_i \rightarrow X$  where each  $U_i$  is a standard étale neighborhood of an affine open subset of  $\dim(R)$ -dimensional affine space over  $\mathbb{k}$ . For any map  $Z \rightarrow X$  define,

$$S_e(Z) = \{\lambda \in (0, 1] \mid \tau(r_0, \dots, r_{q-1}, \lambda', e)|_Z \neq \tau(r_0, \dots, r_{q-1}, \lambda, e)|_Z \forall \lambda' > \lambda\}.$$

By 2.17, it is obvious that  $S_e(\text{id} : X \rightarrow X) = \cup_i S_e(U_i)$ . Therefore if  $\lambda$  is an accumulation point of  $S = \cup_e S_e(\text{id} : X \rightarrow X)$  then, because the set  $\{U_i\}$  is finite,  $\lambda$  is an accumulation point of some  $\cup_e S_e(U_i)$ . If  $S$  has an infinite number of accumulation points, then for some  $i$ ,  $S(U_i) = \cup_e (S_e(U_i))$  must have an infinite number of accumulation points. Therefore it is enough to prove the lemma for each  $U_i$ . That is, we may assume  $X$  is a standard étale neighborhood of an affine open subspace of affine space over  $\mathbb{k}$ .

Therefore, it is enough to prove the theorem when  $R$  is the localization of  $\mathbb{k}[x_1, \dots, x_n][t]/(f(t))$  with  $f(t)$  an irreducible monic polynomial. By 3.11, it is enough to prove the result when  $R$  is the localization of a polynomial ring and, thus, enough to prove the result when  $R$  is a polynomial ring. This is 3.12.  $\square$

It will be useful to have to understand the analytic relationship between the sets  $S_e$  and the accumulation points of  $S$ .

**Theorem 3.15.** If  $R$  is smooth and of essentially finite type over  $\mathbb{k}$  then there exists an integer  $N$  such that for all  $e \geq N$ ,

$$S_e \subset \left\{ \frac{\lceil \lambda q^{e+1} \rceil - a}{q^{e+1}} \mid \lambda \text{ is a jumping number for } r_0, \dots, r_{q-1}, 0 \leq a < q^N \right\}.$$

*Proof.* The arguments from 3.12 imply that there are no infinite sequences  $s_n \in S_{e_n}$  with  $s_{n+1} < s_n$  and  $e_n \geq e_{n+1}$ .

Claim 1: For any  $\epsilon > 0$  such that all the jumping numbers are distance at least  $3\epsilon$  away from each other, there exists  $E_\epsilon$  such that for all  $e \geq E_\epsilon$  and all  $s \in S_e$  there exists a unique jumping number  $\lambda$  with  $0 \leq \lambda - s < \epsilon$ . Note that such an  $\epsilon$  exists because the jumping numbers in  $(0, 1]$  are a finite set.

To prove this claim, we proceed by contradiction. Suppose that for every  $E = n$  we could find  $e_n > E$  such that for some  $s_n \in S_{e_n}$  every jumping number is either more than  $\epsilon$  away from  $s_n$  or is within distance  $\epsilon$  of  $s_n$  but is smaller than  $s_n$ . By passing to a subsequence, we may assume the sequence  $s_n$  is either always in an  $\epsilon$ -neighborhood of a jumping number  $\lambda_0 < s_n$  or is always at least  $\epsilon$  away from any jumping number. By passing to a further subsequence, we may assume  $s_n$  converges in  $[0, 1]$ . Now in the first case, as all jumping numbers are at least distance  $3\epsilon$  apart, it must be that the numbers converge to  $\lambda_0$ . However, this means that  $s_n$  has a strictly decreasing subsequence which can not happen by the proof of 3.12. By invoking the proof of 3.12, it is not possible that the sequence  $s_n$  converges to 0. Thus in the second case,  $s_n$  converges to a jumping number that is at least  $\epsilon$  away from all other jumping numbers. In particular, there is a jumping number at least  $\epsilon$  away from itself. This is also a contradiction.

We claim we can make this statement even stronger.

Choose  $N' \geq 2$  such that any jumping number  $\lambda$  with  $q\lambda$  not an integer is at least  $q^{-N'-1}$  away from any number of the form  $\frac{a}{q}$  with  $0 \leq a \leq q$ .

Claim 2: For every  $\epsilon \in (0, q^{-N'})$  with all jumping numbers at least distance  $3q\epsilon$  apart from each other, 0, and 1 (if it is not already a jumping number) then there exists  $N_\epsilon$  such that for all  $e \geq N_\epsilon$  and  $s \in S_e$  there exists a unique jumping number  $\lambda$  with  $0 \leq \lambda - s < \epsilon q^{-e+N_\epsilon}$ .

Fix  $\epsilon$  as in the hypothesis of claim 2 and choose  $E_\epsilon$  as in claim 1. Set  $N_\epsilon = \max\{E_\epsilon, N'\}$  and proceed by induction on  $e \geq N_\epsilon$ .

Base case: If  $e = N_\epsilon$  then we are done because  $e = N_\epsilon \geq E_\epsilon$  and  $s \in S_e$  implies there exists a unique jumping number  $\lambda$  with  $0 \leq \lambda - s < \epsilon = \epsilon q^{-e+N_\epsilon}$ .

Inductive step: Assume for all  $s \in S_{e-1}$  ( $e - 1 \geq N_\epsilon \geq E_\epsilon$ ) there exists a unique jumping number  $\lambda$  such that  $0 \leq \lambda - s < \epsilon q^{-e+1+N_\epsilon}$ .

Fix  $s \in S_e$ . By claim 1, there exists a unique jumping number  $\lambda'$  with  $0 \leq \lambda' - s < \epsilon$ . We break into cases.

Case 1:  $qs$  is an integer.

Write  $s = \frac{a}{q}$  for some  $0 < a < q^{e+1}$  then we have  $0 \leq q\lambda' - a < q\epsilon$ . Thus,  $0 \leq fr(q\lambda') = fr(q\lambda' - a) \leq q\epsilon$ . Hence,  $q\lambda'$  cannot be a jumping number since all jumping numbers are distance at least  $3q\epsilon$  away from 0. By 3.9 we are left to conclude that  $q\lambda'$  is an integer which is  $q\epsilon$  away from  $a$ . By choice of  $\epsilon < q^{-N'}$  and  $N' \geq 2$ ,  $q\lambda' = a$  and  $\lambda = s$ .

Case 2:  $qs$  is not an integer.

By inductive hypothesis there exists a unique jumping number  $\lambda$  with  $0 \leq \lambda - fr(qs) \leq \epsilon q^{-e+1+N_\epsilon}$ .

It is enough to show that  $s = \lambda' - \frac{\lambda}{q} + fr(qs)/q$  because then the result follows from direct computation.

If  $q\lambda'$  is an integer then,

$$0 \leq fr(q\lambda' - qs) = 1 - fr(qs) \leq q\epsilon$$

implies by choice of  $\epsilon$  that  $\lambda = 1$ . If  $q\lambda'$  is not an integer then by 3.10 and choice of  $\epsilon$ ,  $fr(q\lambda') = \lambda$ . In either situation, we recover that  $q\lambda' - \lambda = a$  for some integer  $a$ .

Hence,

$$-q\epsilon < q\lambda' - \lambda - (qs - fr(qs)) = a - qs + fr(qs) < q\epsilon$$

and the middle term is an integer. Since  $\epsilon < q^{-2}$ ,  $a = qs - fr(qs)$  and we are done.

To finish the proof of this theorem, choose  $\epsilon$  meeting the criteria of claim 2 and choose  $N \geq N_\epsilon$  so that  $q^N > \lceil \epsilon q^{N_\epsilon+1} \rceil$ . For  $e \geq N$  and  $s = \frac{m}{q^{e+1}} \in S_e$ , we have  $sq^{e+1} \leq \lambda q^{e+1} < sq^{e+1} + \epsilon q^{N_\epsilon+1}$ . Thus,

$$\begin{aligned} m &= \lceil sq^{e+1} \rceil \\ &\leq \lceil \lambda q^{e+1} \rceil \\ &\leq \lceil sq^{e+1} \rceil + \lceil \epsilon q^{N_\epsilon+1} \rceil \\ &< m + q^N \end{aligned}$$

We conclude,  $\lceil \lambda q^{e+1} \rceil \geq m > \lceil \lambda q^{e+1} \rceil - q^N$ .

□



### 3.2 List test modules

We will now consider a generalization of simple list test ideals called list test modules.

**Definition 3.16.** Let  $\{A_{k,n}\}_{k,n \geq 0}^{\infty, q^{-1}} \in M_l(R)$  be a double-indexed set of matrices with only finitely many non-zero. These matrices act on the free module  $M = R^{\oplus l}$ . For a matrix  $A$  write  $A^{[q]}$  for the matrix whose entries are the  $q^{th}$  power of the entries of  $A$ . To ease notation, the indexing set will be expanded to all of  $\mathbb{Z} \times \mathbb{Z}$  by setting the undefined indices to be the zero matrix.

Define polynomials valued in  $M_l(R)$  inductively on  $e$  by setting

$$H_n^1(\tau) = \sum_k A_{k,n} \tau^k,$$

$$A(t) = \sum_{0 \leq n < q} H_n^1(t^q) t^n,$$

and defining  $H_n^e(\tau)$  to be the unique polynomials such that

$$A(t)^{[q^{e-1}]} \dots A(t) = \sum_{0 \leq n < q^e - 1} H_n^e(t^{q^e}) t^n.$$

The next lemma shows that there exists a unique minimal  $N$  such that for all  $e$  and  $n$

$$H_n^e(\tau) \in M_n(R) \oplus M_n(R)\tau \oplus \dots \oplus M_n(R)\tau^N.$$

For each  $e \geq 0$  and  $\lambda \in (0, 1]$  define the list test module as

$$\tau(\{A_{k,j}\}, \lambda, e) = \sum_{\lambda' \leq \lambda} (H_{[\lambda' q^{e+1}]-1}^{e+1}(\tau) R^{\oplus l})^{[\frac{1}{q^{e+1}}]} \subset R^{\oplus lN} \cong R^{\oplus l} \oplus R^{\oplus l}\tau \oplus \dots \oplus R^{\oplus l}\tau^N$$

**Lemma 3.17.** The integers  $\deg_{\tau} H_n^e(\tau)$  are uniformly bounded (independently of  $e$  and  $m$ )

*Proof.* For each  $e$ , set

$$M_e = \max_{0 \leq n < q^e} \deg_{\tau} H_n^e(\tau).$$

Recall the definition of the polynomials  $H_n^e(\tau)$ .

$$A(t)^{[q^{e-1}]} \dots A(t) = \sum_{n=0}^{q^e-1} H_n^e(t^{q^e}) t^n$$

If  $A(t)$  has  $t$ -degree at most  $d$  the the left side of this equation has  $t$ -degree at most  $\frac{d(q^e-1)}{q-1}$ . The right side of this equation has  $t$ -degree at least  $q^e M_e$ . This yields the inequality,

$$\frac{d(q^e-1)}{q-1} \geq q^e M_e$$

Dividing by  $q^e$ ,

$$M_e \leq \frac{d}{q-1} - \frac{d}{q^{e+1} - q^e} \leq \frac{d}{q-1}.$$

□

**Example 3.18.** If  $\{A_{k,n}\}$  be a list of matrices with the following properties,

1.  $A_{k,n} = 0$  for all  $k > 0$
2.  $A_{k,n} \in M_1(R) = R$  (i.e.  $l = 1$ )

and we set  $r_i = A_{0,i}$  then  $\tau(\{A_{k,n}\}, \lambda, e) = \tau(r_0, r_1, \dots, r_{q-1}, \lambda, e)$  for all  $\lambda$  and  $e$  where the latter ideal is the simple list test ideal.

*Proof.* First let us prove the formula

$$H_{i_0+i_1q+\dots+i_eq^e}^e(\tau) = r_{i_0}r_{i_1}^q\dots r_{i_e}^{q^e}$$

by induction on  $e$ .

When  $e = 1$  the formula is clear. Suppose the formula is true for  $e$ , that is assume

$$A^{[q^{e-1}]}(t)\dots A(t) = \sum_{i_0+i_1q+\dots+i_{e-1}q^{e-1}} r_{i_0}r_{i_1}^q\dots r_{i_{e-1}}^{q^{e-1}} t^{i_0+i_1q+\dots+i_{e-1}q^{e-1}}$$

This implies

$$A^{[q^e]}(t)\dots A(t) = \sum_{i_0+i_1q+\dots+i_{e-1}q^{e-1}} \sum_{i_e} r_{i_0}r_{i_1}^q\dots r_{i_{e-1}}^{q^{e-1}} t^{i_0+i_1q+\dots+i_{e-1}q^{e-1}} r_{i_e}^{q^e} t^{i_eq^e},$$

which gives the desired result.

Plugging this formula back into the definition and taking  $N = 1$ , we obtain  $\tau(\{A_{k,n}\}, \lambda, e) = \tau(r_0, \dots, r_{q-1}, \lambda, e)$ . □

**Definition 3.19.** Analogously to the simple case, define

$$S_e = \{\lambda | \tau(\{A_{k,n}\}, \lambda, e) \neq \tau(\{A_{k,n}\}, \lambda', e) \forall \lambda' > \lambda\},$$

$S = \cup_e S_e$  and the **set of jumping numbers of  $\{A_{k,n}\}$**  to be the non-zero accumulation points of  $S$ . This definition is extended to all real numbers by defining  $\lambda$  to be a jumping number if some integer translate of  $\lambda$  is a jumping number.

It will be shown in this section that this set of jumping numbers is again discrete and rational. First, it will be necessary to generalize the results from the case of simple list test ideals.

**Proposition 3.20.** If  $\lambda$  is a jumping number for  $\{A_{k,n}\}$  then either  $q\lambda$  is an integer or  $q\lambda$  is also a jumping number for  $\{A_{k,n}\}$ . In particular, if  $q\lambda$  is not an integer then the fractional part of  $q\lambda$  is a jumping number for  $\{A_{k,n}\}$ .

*Proof.* As before we proceed by contraposition and first notice that if we write

$$H_\beta^{e-1}(\tau) = \sum_k B_{k,\beta} \tau^k$$

for any  $0 \leq \beta < q^{e-1}$  then

$$\begin{aligned} A^{[q^{e-1}]}(t) \dots A(t) &= \sum_n \sum_k \sum_\beta (H_n^1)^{[q^{e-1}]}(t^{q^e}) t^{nq^{e-1}} B_{k,\beta} t^{kq^{e-1}} t^\beta \\ &= \sum_j \sum_{n+k=j} \sum_\beta (H_n^1)^{[q^{e-1}]}(t^{q^e}) B_{k,\beta} t^{\beta+jq^{e-1}} \\ &= \sum_{0 \leq j_0 < q} \sum_{j_1} \sum_{n+k=j_0+j_1q} \sum_\beta (H_n^1)^{[q^{e-1}]}(t^{q^e}) B_{k,\beta} t^{\beta+j_0q^{e-1}+j_1q^e} \\ &= \sum_{0 \leq j_0 < q} \sum_{j_1} \sum_n \sum_\beta (H_n^1)^{[q^{e-1}]}(t^{q^e}) B_{j_0+j_1q-n,\beta} t^{\beta+j_0q^{e-1}+j_1q^e} \end{aligned}$$

where  $G^{[q^{e-1}]}(\tau)$  is the polynomial whose coefficients are the  $[q^{e-1}]$  powers of the coefficients of  $G$ .

This implies

$$(*) \quad H_{\beta+j_0q^{e-1}}^e(\tau) = \sum_{j_1} \sum_n (H_n^1)^{[q^{e-1}]}(\tau) B_{j_0+j_1q-n,\beta} \tau^{j_1}.$$

By flatness of Frobenius and 2.20, for any  $e$  and  $0 < \beta < q^{e-1}$ ,

$$\tau(\{A_{k,n}\}, \frac{\beta+1}{q^{e-1}}, e-2) = \tau(\{A_{k,n}\}, \frac{\beta}{q^{e-1}}, e-2)$$

if and only if for every  $v \in R^{\oplus l}$

$$H_\beta^{e-1}(\tau)v \in \sum_{\beta' < \beta} \mathbb{D}_R^{e-1} H_{\beta'}^{e-1}(\tau) R^{\oplus l}.$$

In particular, if  $\tau(\{A_{k,n}\}, \frac{\beta+1}{q^{e-1}}, e-2) = \tau(\{A_{k,n}\}, \frac{\beta}{q^{e-1}}, e-2)$  then there exists  $\{P_{\beta'}\} \subset \mathbb{D}_R^{e-1}$  and  $v_{\beta'} \in R^{\oplus l}$  such that

$$\sum_{\beta' < \beta} P_{\beta'} B_{k,\beta'} v_{\beta'} = B_{k,\beta} v$$

for all  $k$ .

If  $0 < \alpha < q^e$ , we may write  $\alpha = \beta + j_0 q^{e-1}$  with  $0 < \beta < q^{e-1}$ , then

$$H_\alpha^e(\tau)v = \sum_{j_1} \sum_{n=j_0+j_1q} (H_n^1)^{[q^{e-1}]}(\tau) B_{j_0+j_1q-n, \beta} \tau^{j_1} v \quad (1)$$

$$= \sum_{j_1} \sum_{n=j_0+j_1q} (H_n^1)^{[q^{e-1}]}(\tau) \sum_{\beta' < \beta} P_{\beta'} B_{j_0+j_1q-n, \beta'} v_{\beta'} \tau^{j_1} \quad (2)$$

$$= \sum_{\beta' < \beta} P_{\beta'} \sum_{j_1} \sum_{n=j_0+j_1q} (H_n^1)^{[q^{e-1}]}(\tau) B_{j_0+j_1q-n, \beta'} v_{\beta'} \tau^{j_1} \quad (3)$$

$$= \sum_{\beta' < \beta} P_{\beta'} H_{\beta'+j_0q^{e-1}}^1(\tau) \quad (4)$$

where the equality between (1) and (2) follows from (\*) and the equality between (2) and (3) follows because  $P_\beta$  acts on  $R^{\oplus l}$  through the diagonal action on each entry of  $R$  and is linear with respect to  $q^{e-1}$  powers by virtue of being in  $\mathbb{D}_R^{e-1}$ .

Therefore,

$$\tau(\{A_{k,n}\}, \frac{\beta + \gamma q^{e-1} + 1}{q^e}, e-1) = \tau(\{A_{k,n}\}, \frac{\beta + \gamma q^{e-1} + 1}{q^e}, e-1).$$

□

The proofs of the remaining statements follow from direct and obvious modification of the simple list test ideal case and replacing 3.8 with the above proposition. As such, they will not be restated.

**Corollary 3.21.**  $\lambda \in (0, 1]$  is a jumping number for  $\{A_{k,n}\}$  then either the fractional part of  $q\lambda$  is a jumping number or  $\lambda = \frac{a}{q}$  with  $0 \leq a < q$ .

**Theorem 3.22.** If  $\lambda$  is a jumping number in the extended sense then either  $q\lambda$  is a jumping number or  $q\lambda$  is an integer. In particular, if  $q\lambda$  is not an integer then the fractional part of  $q\lambda$  is a jumping number.

**Remark 3.23.** The equation (\*) in 3.20 implies for fixed  $\lambda$  the ideals  $\tau(\{A_{k,n}\}, \lambda, e)$  are decreasing as  $e$  increases.

**Proposition 3.24.** Let  $x_1, \dots, x_n \in R$  be a regular sequence defining a maximal ideal  $m \subset R$  such that

1.  $F_{R*}^{(e+1)}R$  is freely generated over  $R$  by  $x^{\vec{u}}$  (in multi-index notation) for  $0 \leq \vec{u} \leq q^{e+1} - 1$ .
2.  $F_{S*}^{(e+1)}S$  is freely generated over  $S$  by  $\bar{x}^{\vec{v}}$  where  $Q : R \rightarrow S = R/(x_n)$ .

If  $\{A_{k,n}\}$  is a list in  $M_l(S)$  and  $\{\widetilde{A_{k,n}}\}$  are representatives in  $R$  with  $Proj_{x^{\vec{u}}}(a) = 0$  for all  $\vec{u}$  with non-zero  $n^{th}$  component and for any entry  $a$  of  $\widetilde{A_{k,n}}$  then

$$\tau(\{\widetilde{A_{k,n}}\}, \lambda, e) + (x_n) = Q^{-1}(\{A_{k,n}\}, \lambda, e).$$

**Lemma 3.25.** If  $R$  is a polynomial ring then the jumping numbers for  $\{A_{k,n}\}$  in  $(0, 1]$  are finite and rational.

**Theorem 3.26.** If  $R$  is smooth and of essentially finite type over  $\mathbb{k}$  then the set of jumping numbers of  $\{A_{k,n}\}$  are discrete and rational.

**Theorem 3.27.** If  $R$  is smooth and of essentially finite type over  $\mathbb{k}$  then there exists an integer  $N$  such that for all  $e \geq N$ ,

$$S_e \subset \left\{ \frac{\lceil \lambda q^{e+1} \rceil - a}{q^{e+1}} \mid \lambda \text{ is a jumping number for } \{A_{k,n}\}, 0 \leq a < q^N \right\}.$$

## 4 Euler Operators in the Affine Case

### 4.1 The definition of a $b$ -function

Now we set  $S = R[t]$  and we can consider the constructions of section 2 but for  $S$ .

We will now consider the higher Euler operators  $\Theta_i = \partial_i^{[p^{i-1}]} t^{p^{i-1}} \in \text{End}_{p^{i+1}}(S)$ . We also consider the operators  $\theta_i = t^{p^{i-1}} \partial_i^{[p^{i-1}]}$ . Notice  $\Theta_i - \theta_i = 1$ . The operators  $\theta_i$  and  $\Theta_i$  satisfy the Artin-Schreier equation  $x^p - x$  and hence have eigenvalues in  $\mathbb{F}_p$ .

**Definition 4.1.** If  $V$  is a space equipped with commuting actions of  $\{\theta_i\}_{i=1}^{\gamma_e}$ . Define  $0 \neq v \in V$  as an eigenvector of eigenvalue  $0 \leq n < q^e$  if when  $n = \sum_{l=0}^{\gamma_e-1} i_l p^l$  is the base  $p$  expansion then  $v$  is an eigenvector of eigenvalue  $i_{l-1}$  for  $\theta_l$ .

If  $V$  is a space equipped with commuting actions of  $\{\Theta_i\}_{i=1}^{\gamma_e}$ . Define  $0 \neq v \in V$  as an eigenvector of eigenvalue  $0 \leq n < q^e$  if when  $n = \sum_{l=0}^{\gamma_e-1} (q-1-i_l) p^l$  is the base  $p$  expansion then  $v$  is an eigenvector of eigenvalue  $-i_{l-1}$  for  $\Theta_l$  for all  $1 \leq l \leq \gamma_e$ . Note that this definition is compatible with the definition of the eigenspaces for  $\{\Theta_i\}$  given in [Mus09].

While the definition for eigenvector for the collection  $\theta_i$  is natural, we will quickly explain the reason for the slightly non-obvious definition for the collection  $\Theta_i$ .

**Remark 4.2.** By the relation  $\Theta_i - \theta_i = 1$ , any space with commuting actions of  $\{\theta_i\}$  also has natural commuting actions of  $\{\Theta_i\}$ .  $v \in V$  is an eigenvector of eigenvalue  $n = \sum_{l=0}^{\gamma_e-1} (q-1-i_l) q^l$  for  $\{\Theta_i\}$  if and only if it is an eigenvector of eigenvalue  $n$  for  $\{\theta_i\}$ . Therefore, to ease the analysis, we will consider eigenvalues for the collection  $\{\theta_i\}$  instead of  $\{\Theta_i\}$ .

**Definition 4.3.** Let  $q = p^\gamma$ . If  $(\mathcal{M}, F)$  is a unit  $F_S$ -module generated by  $(M, A)$  with  $M$  finitely generated over  $S$ , define a  $b$ -function for  $((\mathcal{M}, F), (M, A))$  to be a polynomial  $b(s) \in \mathbb{R}[s]$  with roots in  $(-1, 0]$  satisfying the following property: If  $\{\lambda_i\}$  denotes the roots of  $b(s-1)$  then there exists an integer  $N$  such that for all  $e \geq 0$ , the set

$$\{\lceil \lambda_i q^e \rceil - a \mid 0 \leq a < q^N\} \cup \{0\}$$

contains all possible eigenvalues for the action of  $\{\theta_i\}$  on

$$\mathbb{D}_R^e[t, \theta_1, \dots, \theta_{\gamma_e}]F_S^{(e-1)*}(A) \dots AM / \mathbb{D}_R^e[t, \theta_1, \dots, \theta_{\gamma_e}]tF_S^{(e-1)*}(A) \dots AM.$$

**Remark 4.4.** This notion of  $b$ -function depends on the root  $(M, A)$  and  $q$ .

**Remark 4.5.** By 2.20, the condition in the definition implies there exists  $N' (= iq^N)$  such that for all  $e$  there are at most  $N'$  non-zero eigenvalues for the collection  $\{\theta_i\}_{i=1}^e$  acting on

$$\mathbb{D}_R^e[t, \theta_1, \dots, \theta_{\gamma_e}]\mu_0(M) / \mathbb{D}_R^e[t, \theta_1, \dots, \theta_{\gamma_e}]t\mu_0(M).$$

In particular, if for each  $0 \leq a < q^N$   $c_{i,j}(a)$  is the  $j$ -th digit in the base  $q$  expansion of  $\lceil \lambda_i q^{e+1} \rceil - a$  then the condition is the same as requiring for all  $j$  that for all  $j$

$$\prod_{0 \leq a < q^N} \prod_i -\Theta_j + 1 + c_{i,j}(a)$$

annihilates

$$\mathbb{D}_R^e[t, \theta_1, \dots, \theta_{\gamma_e}]\mu_0(M) / \mathbb{D}_R^e[t, \theta_1, \dots, \theta_{\gamma_e}]t\mu_0(M).$$

Hence, the condition in the definition is stronger than the condition given in characteristic 0 and is a generalization of the condition given in [Mus09].

**Example 4.6.** Let  $f \in R$ ,  $q = p$ , and  $\mathcal{M} = R[t]_{(f-t)} / R[t] \cong H_{\Gamma_f}^1(\mathcal{O}_{\text{Spec}(R[t])})$  is equipped with the natural unit  $F$ -module as an  $R[t]$ -module.  $(\mathcal{M}, F)$  is generated by  $R[t]/(t) \rightarrow R[t]/(t)$  via multiplication by  $(f - t)^{p-1}$ . The combined results of [Mus09] and 3.4 show that a  $b$ -function for this generator is given by  $\prod (s + \lambda_i)$  where  $\lambda_i$  are the jumping numbers for the generalized test ideals  $\tau(f^\lambda)$ . This is the characteristic  $p$  analogue of the characteristic 0 statement (see [BS05] or [ELSV04]) that if  $\lambda$  is a jumping exponent then  $b(-\lambda) = 0$ .

The next proposition provides a way to simplify the analysis of the existence of  $b$ -functions by reducing to the case when  $M$  is free of finite rank.

**Proposition 4.7.** If  $(\mathcal{M}, F)$  is generated by  $(M, A)$  with  $M$  a finite  $S$ -module then any free resolution  $\pi : S^{\oplus l} \rightarrow M$  gives rise to a commutative diagram,

$$\begin{array}{ccc} S^{\oplus l} & \xrightarrow{\tilde{A}} & (F_S^* S)^{\oplus l} \\ \downarrow \pi & & \downarrow F_S^*(\pi) \\ M & \xrightarrow{A} & F_S^*(M) \end{array}$$

Let  $(\widetilde{M}, \widetilde{F})$  denote the unit  $F$ -module generated by  $(S^{\oplus l}, \tilde{A})$ . If  $(\widetilde{M}, \widetilde{F})$  admits a  $b$ -function then so does  $(\mathcal{M}, F)$ .

*Proof.*

$$F_S^{e*}(\pi)(\mathbb{D}_R^e[t, \theta_1, \dots, \theta_{\gamma_e}]F_S^{(e-1)*}(\tilde{A})\dots\tilde{A}(S^{\oplus l})) = \mathbb{D}_R^e[t, \theta_1, \dots, \theta_{\gamma_e}]F_S^{(e-1)*}(A)\dots AM$$

and

$$F_S^{e*}(\pi)(\mathbb{D}_R^e[t, \theta_1, \dots, \theta_{\gamma_e}]tF_S^{(e-1)*}(\tilde{A})\dots\tilde{A}(S^{\oplus l})) = \mathbb{D}_R^e[t, \theta_1, \dots, \theta_{\gamma_e}]F_S^{(e-1)*}(A)\dots AM$$

Thus the quotient of the left equations surjects onto the quotient of the right equations and the same  $b$ -function for  $(\tilde{M}, \tilde{F})$  will work for  $(M, F)$ .  $\square$

## 4.2 The freely generated case

**Notation 4.8.** If  $A : S^{\oplus l} \rightarrow S^{\oplus l}$  and consider  $A$  as a matrix of  $R$ -valued polynomials in  $t$ . We can also consider the composition  $F_S^{(e-1)*}(A)\dots A$  as a matrix of polynomials. It will be convenient to decompose these expressions in terms of the degrees of  $t$  between 0 and  $q^e - 1$ . Define  $H_n^e(\tau) \in M_l(R[\tau])$  in the following manner,

$$F_S^{(e-1)*}(A)\dots A = \sum_{0 \leq n < q^e} H_n^e(t^{q^e})t^n.$$

**Remark 4.9.** Lemma 4.13 will show that this choice of notation  $H_n^e(\tau)$  does not conflict with the notation from section 3.

**Lemma 4.10.**

$$\mathbb{D}_R^e[t, \theta_1, \dots, \theta_{\gamma_e}]F_S^{(e-1)*}(A)\dots AS^{\oplus l} = \sum_{0 \leq n < q^e} \mathbb{D}_R^e[t]H_n^e(t^{q^e})t^n R^{\oplus l}$$

*Proof.* To show that

$$\mathbb{D}_R^e[t, \theta_1, \dots, \theta_{\gamma_e}]F_S^{(e-1)*}(A)\dots AS^{\oplus l} \supset \sum_{0 \leq n < q^e} \mathbb{D}_R^e[t]H_n^e(t^{q^e})t^n R^{\oplus l},$$

consider that for each  $0 \leq n < q^e$ , the Euler operators  $t^n \partial_t^{[n]}$  are in the ring generated by  $\{\theta_i\}_{i=1}^e$  and  $\mathbb{F}_p$ . These operators commute with  $t^{q^e}$ . For any  $v \in R^{\oplus l}$ ,  $H_{q^e-1}^e(t^{q^e})t^{q^e-1}v$  is in  $\mathbb{D}_R^e[t, \theta_1, \dots, \theta_{\gamma_e}]F_S^{(e-1)*}(A)\dots AS^{\oplus l}$  because it is equal to  $t^{q^e-1} \partial_t^{[q^e-1]}$  applied to  $F_S^{(e-1)*}(A)\dots Av$ . Notice that if  $m$  is an integer and  $\{H_n^e(t^{q^e})t^n v\}_{n > m}$  is in  $\mathbb{D}_R^e[t, \theta_1, \dots, \theta_{\gamma_e}]F_S^{(e-1)*}(A)\dots AS^{\oplus l}$ , then so is  $H_m^e(t^{q^e})t^m v$  because it is a non-zero scalar multiple of the trailing term of  $t^m \partial_t^{[m]} F_S^{*e}(A)\dots Av$  and the higher terms are in  $\mathbb{D}_R^e[t, \theta_1, \dots, \theta_{\gamma_e}]F_S^{(e-1)*}(A)\dots AS^{\oplus l}$ . The well-ordering principle then proves the containment.

For the opposite containment, it is enough to show that  $\sum_{0 \leq n < q^e} \mathbb{D}_R^e[t]H_n^e(t^{q^e})t^n R^{\oplus l}$  is a left  $\mathbb{D}_R^e[t, \theta_1, \dots, \theta_{\gamma_e}]$ -module because it clearly contains  $F_S^{*e}(A)\dots AS^{\oplus l}$ . It is only required to show that it is closed under multiplication by  $\theta_i$ . By linearity, it is enough to check for each  $i, n, d$ , and  $v$  that  $\theta_i t^d H_n^e(t^{q^e})t^n v$  is contained in  $\sum_{0 \leq n < q^e} \mathbb{D}_R^e[t]H_n^e(t^{q^e})t^n R^{\oplus l}$ . This is clear.  $\square$

**Theorem 4.11.** Consider a unit  $F_S$ -module  $(\mathcal{M}, F)$  generated by  $A : S^{\oplus l} \rightarrow S^{\oplus l}$ . For all  $e$  and for  $0 \leq m < q^e$ ,

$$\mathbb{D}_R^e H_0^e(\tau) R^{\oplus l} + \dots + \mathbb{D}_R^e H_{m-1}^e(\tau) R^{\oplus l} = \mathbb{D}_R^e H_0^e(\tau) R^{\oplus l} + \dots + \mathbb{D}_R^e H_m^e(\tau) R^{\oplus l}$$

as subsets of  $R[\tau]^{\oplus l}$  implies

$$(\mathbb{D}_R^e[t, \theta_1, \dots, \theta_e] F_R^{(e-1)*}(A) \dots AS^{\oplus n})_m = (\mathbb{D}_R^e[t, \theta_1, \dots, \theta_e] t F_R^{(e-1)*}(A) \dots AS^{\oplus n})_m$$

where subscript  $m$  denotes taking the  $m$ -th eigenspace for the operators  $\{\theta_i\}$ .

*Proof.* By the previous lemma,

$$\mathbb{D}_R^e[t, \theta_1, \dots, \theta_e] F_S^{(e-1)*}(A) \dots AS^{\oplus l} = \sum_{0 \leq n < q^e} \mathbb{D}_R^e[t] H_n^e(t^{q^e}) t^n R^{\oplus l}$$

Let  $x \in (\mathbb{D}_R^e[t, \theta_1, \dots, \theta_{\gamma_e}] F_S^{(e-1)*}(A) \dots AS^{\oplus l})_m$  and write

$$x = \sum_{0 \leq n < q^e} \sum_i \sum_{0 \leq k < q^e} P_{k,n,i}(t^{q^e}) t^k H_n^e(t^{q^e}) t^n v_{n,i}$$

where  $P_{k,n,i}(t^{q^e}) \in \mathbb{D}_R^e[t^{q^e}]$  and  $v_{n,i} \in R^{\oplus l}$ .

We may rewrite this sum as

$$x = \sum_i \sum_{0 \leq u < 2q^e - 1} \sum_{0 \leq n < q^e} P_{u-n,n,i}(t^{q^e}) H_n^e(t^{q^e}) t^u v_{n,i}.$$

By virtue that  $x$  lives in the  $m^{th}$  eigenspace

$$x = \sum_i \left( \sum_{0 \leq n \leq m} P_{m-n,n,i}(t^{q^e}) H_n^e(t^{q^e}) t^m v_{n,i} + \sum_{m < n < q^e} P_{m+q^e-n,n,i}(t^{q^e}) H_n^e(t^{q^e}) t^{m+q^e} v_{n,i} \right).$$

For  $n < m$ ,  $H_n^e(t^{q^e}) t^m v_n = t^{m-n} H_n^e(t^{q^e}) t^n v_n$  is in  $(\mathbb{D}_R^e[t, \theta_1, \dots, \theta_{\gamma_e}] t F_S^{(e-1)*}(A) \dots AS^{\oplus l})_m$ . Also,  $\sum_{m < n < q^e} P_{m+q^e-n,n,i}(t^{q^e}) H_n^e(t^{q^e}) t^{m+q^e} v_{n,i}$  is in  $(\mathbb{D}_R^e[t, \theta_1, \dots, \theta_{\gamma_e}] t F_S^{(e-1)*}(A) \dots AS^{\oplus l})_m$ .

Thus it is enough to show that the  $m^{th}$  term  $\sum_i P_{0,m,i}(t^{q^e}) H_m^e(t^{q^e}) t^m v_{m,i}$  is in  $(\mathbb{D}_R^e[t, \theta_1, \dots, \theta_{\gamma_e}] t F_S^{(e-1)*}(A) \dots AS^{\oplus l})_m$ .

Write  $P_{0,m,i}(t^{q^e}) = \sum_j P_{j,i} t^{jq^e}$  where  $P_{j,i} \in \mathbb{D}_R^e$ . By assumption, for each  $i$  there exist  $Q_{j,w,i} \in \mathbb{D}_R^e$  and  $v_{j,w,i} \in R^{\oplus l}$  such that  $\sum_{0 \leq w < m} Q_{j,w,i} H_w^e(\tau) v_{j,w,i} = P_{j,i} H_m^e(\tau) v_{m,i}$ . Setting  $\tau = t^{q^e}$ , multiplying both sides by  $t^{jq^e} t^m$ , and summing over  $i, j$  we obtain

$$\begin{aligned} \sum_i \sum_j \sum_{0 \leq w < m} Q_{j,w,i} t^{jq^e} t^{m-w} H_w^e(t^{q^e}) t^w v_{j,w,i} &= \sum_i \sum_j P_{j,i} t^{jq^e} H_m^e(t^{q^e}) t^m v_{m,i} \\ &= \sum_i P_{0,m,i}(t^{q^e}) H_m^e(t^{q^e}) v_{m,i}. \end{aligned}$$



The left side of the above equation is contained in  $(\mathbb{D}_R^e[t, \theta_1, \dots, \theta_{\gamma_e}]tF_S^{(e-1)*}(A)\dots AS^{\oplus l})_m$  as the sum is over  $w < m$ , thereby completing the proof.  $\square$

**Corollary 4.12.** If there is a non-zero eigenvector of weight  $m$  for the operators  $\{\theta_i\}$  in

$$\mathbb{D}_R^e[t, \theta_1, \dots, \theta_e]F_S^{(e-1)*}(A)\dots AS^{\oplus l} / \mathbb{D}_R^e[t, \theta_1, \dots, \theta_e]tF_S^{(e-1)*}(A)\dots AS^{\oplus l}$$

then

$$\mathbb{D}_R^e H_0^e(\tau)R^{\oplus l} + \dots + \mathbb{D}_R^e H_{m-1}^e(\tau)R^{\oplus l} \neq \mathbb{D}_R^e H_0^e(\tau)R^{\oplus l} + \dots + \mathbb{D}_R^e H_m^e(\tau)R^{\oplus l}.$$

### 4.3 Relationship to list test modules

In this subsection, we continue working in the context of the previous subsection. We will show, similar to the case of the first local cohomology module, that the (infinite) behavior of the eigenvalues of  $\{\theta_i\}$  is completely controlled by the jumping numbers of list test modules when  $R$  is smooth and of essentially finite type over  $\mathbb{k}$ . First, we begin by relating the eigenvalues to the sets  $S_e$ .

**Lemma 4.13.** If  $(\mathcal{M}, F)$  is generated by  $(M, A(t))$ ,  $H_n^1(\tau) = \sum_k A_{k,n} \tau^k$ , and  $S_e$  is the set associated to  $\{A_{k,n}\}$  in section 2, then the eigenvalues of the action of  $\{\theta_i\}_{i=1}^{\gamma_e}$  on

$$\mathbb{D}_R^e[t, \theta_1, \dots, \theta_{\gamma_e}]F_S^{(e-1)*}(A)\dots AS^{\oplus l} / \mathbb{D}_R^e[t, \theta_1, \dots, \theta_{\gamma_e}]tF_S^{(e-1)*}(A)\dots AS^{\oplus l}$$

are contained in the set  $\{m | \frac{m}{q^e} \in S_{e-1}\} \cup \{0\}$ .

*Proof.* In the canonical basis for  $S^{\oplus l}$ ,  $F^{e*}A = A(t)^{[q^e]}$  in the notation of section 3). It then follows directly from the definition that both constructions yield the same definitions for  $H_n^e(\tau)$ .

By 4.12, if there is a non-zero eigenvector of weight  $m \neq 0$  in

$$\mathbb{D}_R^e[t, \theta_1, \dots, \theta_{\gamma_e}]F_S^{(e-1)*}(A)\dots AS^{\oplus l} / \mathbb{D}_R^e[t, \theta_1, \dots, \theta_{\gamma_e}]tF_S^{(e-1)*}(A)\dots AS^{\oplus l}$$

then

$$\mathbb{D}_R^e H_0^e(\tau)R^{\oplus l} + \dots + \mathbb{D}_R^e H_{m-1}^e(\tau)R^{\oplus l} \neq \mathbb{D}_R^e H_0^e(\tau)R^{\oplus l} + \dots + \mathbb{D}_R^e H_m^e(\tau)R^{\oplus l}$$

which implies

$$\tau(\{A_{k,n}\}, \frac{m}{q^e}, e-1)^{[q^e]} \neq \tau(\{A_{k,n}\}, \frac{m+1}{q^e}, e-1)^{[q^e]}.$$

By the faithful flatness of Frobenius, the last inequality is true if and only if  $\frac{m}{q^e} \in S_{e-1}$ .  $\square$

The next theorem explicitly describes how the jumping numbers control the (infinite) behavior of the eigenvalues of the operators  $\theta_i$  when  $X$  is of essentially finite type over  $\mathbb{k}$ .

**Theorem 4.14.** If  $X$  is of essentially finite type over  $\mathbb{k}$  and  $(\mathcal{M}, F)$  is generated by  $(M, A(t))$  then there exists  $N$  such that for all  $e > 0$  the eigenvalues of the action of  $\{\theta_i\}_{i=1}^{\gamma_e}$  on

$$\mathbb{D}_R^e[t, \theta_1, \dots, \theta_{\gamma_e}]F_S^{(e-1)*}(A) \dots AS^{\oplus l} / \mathbb{D}_R^e[t, \theta_1, \dots, \theta_{\gamma_e}]tF_S^{(e-1)*}(A) \dots AS^{\oplus l}$$

are contained in the set

$$\{[\lambda q^{e+1}] - a | \lambda \text{ is a jumping number for } \{A_{k,n}\}, 0 \leq a < q^N\} \cup \{0\}$$

where  $H_n^1(\tau) = \sum_k A_{k,n} \tau^k$ .

*Proof.* Combine 3.27 with the previous lemma. □

## 4.4 The Lemma on $b$ -functions in positive characteristic

**Theorem 4.15.** (The lemma on  $b$ -functions) If  $X$  is smooth and of essentially finite type over  $\mathbb{k}$  and  $(\mathcal{M}, F)$  is a locally finitely generated unit  $F$ -module on  $X \times \mathbb{A}^1$  then for every affine open set  $U \subset X$  a  $b$ -function with rational roots exists for every generator  $(M, A)$  of  $(\mathcal{M}|_{U \times \mathbb{A}^1}, F|_{U \times \mathbb{A}^1})$ .

*Proof.* Follows from 4.14. □

## 4.5 Examples

**Example 4.16.** (Free cyclic generators of low degree) If  $l = 1$  and each  $H_n^1(\tau)$  is constant, then a  $b$ -function for  $(\mathcal{M}, F)$  is given taking the polynomial whose roots are the jumping numbers for the list  $H_0^1, \dots, H_{q-1}^1$ . This follows directly from 3.18.

**Example 4.17.** (The free resolution of the first local cohomology module) Let  $f \in R$  then the first local cohomology module is  $F_S$ -generated in the sense of G. Lyubeznik by

$$R = S/tS \rightarrow S/tS = R \text{ by } \overline{p(t)} \mapsto (f - t)^{p-1} \overline{p(t)}.$$

It has free resolution  $S \rightarrow S$  by multiplication by  $(f - t)^{p-1}$ .

Combining the previous example with 3.4, one computes that a  $b$ -function for the first local cohomology module of  $f$  (and its free resolution) is given by taking the polynomial whose roots are the negatives of  $F$ -jumping exponents of  $f$ . The stronger result of [Mus09] both coincides with this result and strengthens it: This is the smallest such polynomial satisfying the criteria to be a  $b$ -function. As previously mentioned, this is analogous to the situation in characteristic 0 where the roots are negatives of jumping exponents for multiplier ideals.

**Example 4.18.** (Pushforward of rank one tame local systems onto  $\mathbb{A}^1$ ) Let  $X = \mathbb{A}^1 = \text{Spec}(\mathbb{k}[t])$ ,  $U = \text{Spec}(\mathbb{k}[t, t^{-1}])$ , and  $m$  an integer dividing  $q - 1$ . Consider the  $\mathbb{k}[t]$ -module  $\mathcal{M} = \mathbb{k}[t, t^{-1}] \sqrt[m]{t}$  and its obvious Frobenius map  $F(f \sqrt[m]{t}) = f^q t^{\frac{q-1}{m}} \sqrt[m]{t}$ . This gives  $\mathcal{M}$

the structure of a unit  $F$ -module on  $X$ . Moreover, one can check that for any  $j \geq 1$   $\mathcal{M}$  is generated as a unit  $F$ -module by the morphism  $\mathbb{k}[t] \rightarrow \mathbb{k}[t]$   $f \mapsto ft^{j(q-1)-\frac{q-1}{m}}$ . This generating morphism is the same as choosing the  $\mathbb{D}_X$ -module generator  $t^{-j} \sqrt[m]{t}$  as  $\mu_0(\mathbb{k}[t]) = \mathbb{k}[t]t^{-j} \sqrt[m]{t}$ . To ease notation, set  $\beta_e = j + \frac{q^e-1}{m}$  then for all  $e \gg 0$ ,

$$H_n^e(\tau) = \begin{cases} 0 & \text{if } n \neq q^e - \beta_e \\ \tau^{(j-1)} & \text{if } n = q^e - \beta_e \end{cases}$$

Therefore, either using 4.13 or the theory developed for simple list test ideals,  $S_e = \{\frac{q^e - \beta_e}{q^e}\}$ . In particular, it matches our characteristic zero intuition that the Euler operators,  $\theta_i$  act semi-simply with a single eigenvalue  $-j + \frac{1}{m}$ . This implies the Euler operators  $-\Theta_i$  act semi-simply with a single eigenvalue  $-1 + j - \frac{1}{m}$ . Notice that in characteristic 0,  $(-\partial_t t - j + 1 + \frac{1}{m})t^{-1} \sqrt[m]{t} = 0$  and so the Euler operator actions coincide across characteristics before letting  $e$  tend to  $\infty$ .

Let us now investigate the behavior of  $b$ -functions across characteristics. The limit point of the sets  $S_e$  is clearly  $(1 - \frac{1}{m})$  and so the  $b$ -function is given by  $(s + \frac{1}{m})$ . Notice that this is independent of  $j$ . This shows the non-triviality of the hypothesis that “there exists an integer  $N$ ” in the definition of a  $b$ -function when the degree of  $A(t)$  becomes larger than  $q$ . The  $b$ -function in characteristic 0 is  $(s - j + 1 + \frac{1}{m})$  and does depend on  $j$ . The  $b$ -functions coincide when  $j = 1$ .

**Example 4.19.** (Pushforward of the wildly-ramified Artin-Schreier local system onto  $\mathbb{A}^1$ ) Let  $X = \mathbb{A}^1 = \text{Spec}(\mathbb{k}[t])$ ,  $U = \text{Spec}(\mathbb{k}[t, t^{-1}])$ , and  $\mathcal{M} = \mathbb{k}[t, t^{-1}, u]/(u^q + tu^{q-1} - t)$  considered as a  $\mathbb{k}[t]$ -module with obvious Frobenius. This Frobenius makes  $\mathcal{M}$  into a unit  $F$ -module over  $X$ . It is the pushforward of a local system that is wildly ramified at the origin. A  $b$ -function is given by  $\prod_{0 \leq a < q} (s + \frac{a}{q})$ .

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